DAHLGREN DIVISION
NAVAL SURFACE WARFARE CENTER
Dahlgren, Virginia 22448-5100

NSWCDD/TR-94/97

STATISTICAL STUDIES OF THE MONOPULSE RATIO

BY GORDON W. GROVES WILLIAM D. BLAIR
SYSTEMS RESEARCH AND TECHNOLOGY DEPARTMENT

OCTOBER 1994

DEPARTMENT OF THE NAVY
NAVAL SURFACE WARFARE CENTER
DAHLGREN, VIRGINIA 22448-5000

WHITE OAK
10901 NEW HAMPSHIRE AVE.
SILVER SPRING, MD 20903-5000
(301) 394-

DAHLGREN, VA 22448-5000
(703) 663- 8921

IN RELY REFER TO:
E282:rpg
Change 1

To all holders of NSWCDD/TR-94/97
Title: Statistical Studies of the Monopulse Ratio

7 December 1994

68 page(s)

This publication is changed as follows:

Replace the secondary distribution statement on the front cover and on the Standard Form 298 (Rev 2-89) from Distribution Statement B with Distribution Statement A (Approved for public release; distribution is unlimited).

Change the number of copies in the Distribution List for Defense Technical Information Center (from 2 copies to 12 copies) and add the following DOD Activity (Conus):

ATTN E29L (TECHNICAL LIBRARY) 1 copy
COMMANDING OFFICER
CSSDD NSWC
6703 W HWY 98
PANAMA CITY FL 32407-7001

Add the following Non-DOD Activities (Conus):

THE CNA CORPORATION 1 copy
PO BOX 16268
ALEXANDRIA VA 22302-0268

ATTN GIFT AND EXCHANGE DIVISION 4 copies
LIBRARY OF CONGRESS
WASHINGTON DC 20540

Insert this change sheet directly behind the front cover in your copy.
Write on the cover "Change 1 inserted"

Approved by:

D. B. Colby

D. B. COLBY, Head

FOREWORD

Fast Antiship Cruise Missiles (ASCMs) represent one of the most severe threats to surface ships of the Navy. The radar tracking of low-flying ASCMs is made difficult by the multipath phenomenon. The random nature of diffuse reflections from the sea surface and internal thermal noise of the receiver produce errors that severely degrade the captured signal. The diffuse reflections result in large errors in the monopulse measurements by the threat's elevation angle. This report describes a study of the effects of the various sources of noise on angle measurements by monopulse radar. Quantifying the dependence of the accuracy of the angle measurements on the noise is very important. The monopulse ratio is generally a complex quantity because of arrivals of reflected signals from different directions. Thus, the bivariate density distribution of the monopulse measurements along both the real and imaginary axes is of interest.

The present study was supported through the Office of Naval research, in part by the Independent Research Program of the Naval Surface Warfare Center, Dahlgren Division, managed by H. Raymond Riedl (R04), and in part by the Surface Weapons Systems Technology Block Program, managed at NSWCDD by Robin Staton. This technical report was reviewed by Dr. Ronald Helmick (B32).

Approved by:

D.B. Colby

D. B. COLBY, Head
Systems Research and Technology Department

ACKNOWLEDGMENTS

The authors are grateful to Dr. Winston Chow and Dr. Ronald Helmick for reading a draft, pointing out deficiencies, and suggesting improvements in an earlier version of this report. The authors are also grateful to John E. Gray for suggesting an effective way to evaluate an important integral needed in the development of some of the density functions. The authors would also like to acknowledge the help of Robert Jones, who performed some computations and checking while at NSWCDD on the Department of Defense Science and Engineering Apprentice Program (SEAP) during the summer of 1994.

ABSTRACT

The accuracy with which the angular position of an object can be measured by monopulse radar depends on the signal level relative to that of the various noise sources. The signals captured in the sum and difference channels of a monopulse system are generally represented by complex phasors because the returning signals can arrive from different directions with different phases. The primary variables consisting of the four real quantities which define these phasors are stochastic variables because of the noise. Consequently, the Monopulse Ratio (MR), which is the difference phasor divided by the sum phasor, is also a stochastic variable because it depends on these measured quantities. Gaussian noise in the sum and difference channels does not usually produce a MR with a Gaussian distribution.

Straightforward statistical methods are used to obtain the joint density of the real and imaginary parts of the MR for several cases under the assumption that the primary variables have a multivariate Gaussian distribution. Three cases of noise correlation in the sum and difference channels are discussed, and the corresponding joint MR densities are developed. These represent a quantitative description of the effects of noise occurring in the primary channels. The marginal density representing the real part of the MR, which has figured prominently in the literature, is displayed in a few simple cases. The results of Monte Carlo experiments are given to show consistency with analytically determined density functions and to demonstrate empirical derivation of the density functions in cases where an analytical approach is not feasible.

The variance of the MR determined by a single observation is calculable (*i.e.,* finite) only if the long tails of the density function are somehow limited. Density functions with finite variance are obtained if a threshhold is placed on the magnitude of the sum phasor to discard returns lying below the threshhold. The resulting variance is developed for two simple cases where the primary mean values are zero. The dependence of the joint MR density on a threshhold value is studied.

CONTENTS

ILLUSTRATIONS

CHAPTER 1

INTRODUCTION

An amplitude-comparison monopulse radar system measures target elevation (or azimuth) by using two antenna lobes which are squinted (*i.e.,* pointed in slightly different directions) relative to each other. The system emits a pulse through both lobes, and combines the returning signal received in the two lobes into sum and difference channels. A target situated on boresight (*i.e.,* midway between the two lobe axes) will reflect identical signals into the two lobes resulting in zero output in the difference channel. For a target off the radar boresight, the output in the difference channel will not be zero. The output of the sum channel provides a reference to which the amplitude and phase in the difference channel can be related. In the absence of glint, multipath, jammers and multiple targets, this relative phase will be zero or 180 degrees depending on which side of boresight the target lies. The amplitude of the difference-channel voltage relative to that of the sum channel determines angular distance from boresight if the target is detected within the radar beam.

In practice, the relation of the difference signal to the sum signal is represented by means of the Monopulse Ratio (MR), which is the ratio of the difference phasor to the sum phasor. In the ideal situation described in the previous paragraph the MR is real, but in actual situations the MR is complex because the returning signals often arrive from different directions with different phases within the main beam of the radar. The MRs in the azimuth and elevation directions are converted to angles in these directions, often considering only the real parts of the MRs.

Several situations give rise to the directional spread of the arriving signals. The most troubling and severe situation arises when a target is close enough to the sea surface that both the target and its image lie within the main radar antenna beam. This is the well-known multipath problem, which has received considerable attention over the years without a sufficiently accurate method of determining target elevation having been found. Multiple targets and jamming also produce signals arriving from different directions. Signals also arrive within a significantly-wide cone of directions from a single extended target (e.g., glint), even in the absence of multipath. All these phenomena result in a phase difference between the sum and difference channels of the monopulse radar that produces a MR which is complex.

Originally, these phenomena were handled by determining target elevation and azimuth from the real part of the MR only. The imaginary part, which arises because of the non-

vanishing phase differences of the arriving signals, was considered an unavoidable nuisance. However, in recent years the imaginary part of the MR has been used to obtain more accurate measurements of target angle.

The primary variables, consisting of the four real quantities which represent the phasors, are stochastic variables because of random errors resulting from the noise. Consequently, the MR, which depends on these measured quantities, is also a stochastic variable. It is of interest to consider how the distribution of the empirically-determined MR depends on the noise sources. The noise, particularly that component resulting from diffuse sea-reflection, is difficult to quantify. The effects of noise having a Gaussian density with specified means and variances are studied. This assumption permits the calculation of the resulting MR density resulting from specific variance models. In view of the uncertainties, especially in the mechanism of reflection from the rough sea surface, it does not seem worthwhile to treat a more complete model.

The real and imaginary parts of the monopulse ratio can be expressed in terms of the real and imaginary parts of the sum and difference phasors by the relation

$$p + iq = \frac{x + iy}{u + iv} \tag{1.1}$$

where $p+iq$ is the complex MR, $x+iy = D$ is the difference-channel phasor, and $u+iv = S$ is the sum-channel phasor. The x and u are commonly referred to as the in-phase components, and the y and v as the quadrature components. The analysis will be done exclusively in the real domain. Accordingly, the relations

$$p = \frac{xu + yv}{u^2 + v^2} \qquad \text{and} \qquad q = \frac{yu - xv}{u^2 + v^2} \tag{1.2}$$

define the stochastic variables p and q, whose distribution is sought, while the distribution of the primary quantities x, y, u, v will be postulated to have a multivariate normal distribution with specified means and covariance matrix. Accordingly, the x, y, u, v variables are allowed to have nonzero means and a general covariance matrix.

Since the mapping is four (*i.e.*, primary real quantities) onto two (*i.e.*, secondary real quantities), a multiplicity of density functions of the (x, y, u, v) variables will result in the same density function for (p, q). The ratio (1.1) is not affected if both numerator and denominator are scaled by the same complex quantity. Accordingly, the MR is determined solely by the magnitude ratio and phase difference of D relative to S. The formulation is sometimes represented as a dependence of the MR on three primary quantities, which might be chosen to be the D to S relative phase and the two magnitudes. It is usually not convenient to reduce the number of primary variables further by replacing the two magnitudes by their ratio because the magnitude of S is a useful indicator of SNR, which is often used as a criterion for acceptance or rejection of a measurement.

The relationship between members of the set of (x, y, u, v) distributions which result in a given (p, q) distribution might be characterized as a "rotational" and a "magnitude" ambiguity. This is because the (p, q) variables retain only the phase and amplitude relations of D to S. This set contains many distributions that may appear unrelated to each other.

The statistical properties of the MR have been considered by Kanter [1], Tullsson [2], Seifer [3] and [4], and others. Most of the previous studies have calculated the density function of real(MR) and/or statistical parameters of the distribution. Kanter, in a remarkable study [1], used sophisticated techniques to obtain the density function of p under quite general conditions for partially-correlated isotropic noise, as is considered here in Chapter 5, but he did not discuss the joint distribution of p and q. Tullsson [2] used the method of conditional probabilities to obtain density functions of both p and q for the same noise correlation model under restrictive assumptions regarding the mean values of the sum and difference phasors, but did not discuss the joint MR density. Both Kanter and Tullson studied the case where the MR is obtained by averaging over n observations and demonstrated that the variance exists only for $n > 1$. Seifer [3] and [4] in a remarkable achievement has been able to calculate the mean value and covariance of the MR under various conditions.

The joint distribution of p and q is important in systems that use both the real and imaginary parts of the MR to determine target position. This report considers the bivariate density function of the monopulse ratio (*i.e.,* of p and q) determined from one trial $n = 1$, under somewhat more general conditions, and determines this density function in terms of elementary functions using a straightforward, elementary method to be described.

The results of this report are consistent with Kanter's and Tullsson's general conclusions, and in a few easily-checked special cases their expressions are shown to agree with those in this report. Monte Carlo simulations were carried out to check some of the analytically-determined density functions, and to obtain the density function in cases that are difficult to examine by analysis. One notable feature is the presence of a "hole" or minimum in the joint density, which occurs for some values of the means of the sum and difference phasors and noise covariance.

The MR is a stochastic variable because of the presence of various types of noise: thermal or electronic noise in the circuitry, thermal background radiation from the sky which is often less important than the other sources, diffuse sea reflection noise, glint, atmospheric variation, etc., all of which contribute to the variance in the distribution of the MR. Most statistical studies of noise, including this one, assume that the effect of the noise on S and D is modeled by adding a Gaussian error to the true value. This is a justifiable assumption for many types of noise, while in other cases it is less appropriate but used anyway because more detailed treatments are difficult or not feasible. Noise also can affect the mean values $\bar{S}$ and $\bar{D}$.

Electronic noise and the diffuse sea reflections add many small random increments to the received signals and thus are often taken to be Gaussian. Atmospheric variation and glint are more complicated, and are not representable as an additive white Gaussian noise. The well-known Swerling models specify the statistical distribution of the voltage amplitude factors for either S or D, and their correlation in time. The distribution of phases depends critically on the details of the glint model, and while studies of the effect of glint on the amplitude of signals received have been reported, to our knowledge very little attention has been given to the effect on the phases.

Seifer [3] has treated a uniformly-distributed phase uncertainty, which leads to the conclusion that whatever values $\bar{S}$ and $\bar{D}$ would have in the absence of the phase uncertainty, these values become zero if it is present. However, meaningful values of the Monopulse Ratio D/S are obtained because the phase difference between S and D is not uniformly distributed and in fact peaks at a definite value. This report describes cases in which $\bar{S}$ and $\bar{D}$ are allowed to have arbitrary values, which correspond to cases where the phase uncertainty does not dominate. One application of this more general approach would be the study of the MR acquired from a fixed target.

The study in this report considers correlated Gaussian noise in the sum and difference channels. It is clear that these assumptions are not valid in all possible noise environments. While no claim is made as to the universal applicability of the Gaussian assumption, the results obtained on this basis are useful in a wide range of noise environments. The statistical method used in the study is outlined in Chapter 2. Monte Carlo methods of studying the statistical distributions are discussed in Chapter 3. The results for uncorrelated isotropic noise are treated in Chapter 4, while partially correlated isotropic noise is treated in Chapter 5. The results for arbitrary Gaussian noise are given in Chapter 6. The exclusion of measurements when the magnitude of S is below a given threshhold is considered in Chapter 7. An overall discussion of the results is presented in Chapter 8. Note that the figures are placed following the text at the end of each chapter.

CHAPTER 2

A CONVENIENT TRANSFORMATION

The development of the MR density function is facilitated by a well-known theorem in statistics, which states that the density distribution of a set of n variables functionally related in a one-to-one manner with a primary set of n variables whose density function is known is obtained simply by substituting the functional relations into the known density distribution and multiplying by the Jacobian of the transformation (see, for example, [5]). As the density function of only two variables, p and q, as defined in (1.1) and (1.2), is sought, the immediate necessity is to define an additional pair of dummy variables in order to obtain a transformation $R^4 \rightarrow R^4$ that is one-to-one and onto. It is convenient to let $r = u$ and $s = v$ be the dummy variables. The direct transformation is then

$$p = \frac{xu + yv}{u^2 + v^2}, \qquad q = \frac{yu - xv}{u^2 + v^2}, \qquad r = u, \qquad s = v \tag{2.1}$$

while the inverse transformation is

$$x = pr - qs, \qquad y = ps + qr, \qquad u = r, \qquad v = s \tag{2.2}$$

It is seen that this transformation is indeed one-to-one over the infinite four-dimensional space. The Jacobian of the transformation is given by

$$\left| \frac{\partial(x, y, u, v)}{\partial(p, q, r, s)} \right| = \begin{vmatrix} r & -s & p & -q \\ s & r & q & p \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = r^2 + s^2 \tag{2.3}$$

This formulation is convenient for the case where there is a threshhold R_0 on the sum phasor modulus. If the joint distribution of u and v is modified to exclude values $u^2 + v^2 < R_0^2$ (which would require renormalization of the density function), the corresponding marginal distribution of p and q is obtained by restricting r and s to values $r^2 + s^2 \geq R_0^2$.

Seven cases will be considered in order of increasing complexity. Chapter 4 considers the first four cases in which the noise in the sum and difference channels is assumed uncorrelated and isotropic. In case I the means of the sum and difference phasors are taken to be zero. In case II the same assumptions are made, except that the sum channel is allowed to have a specified nonzero mean. Case III is the same as Case II, except that only the difference phasor is allowed to have a specified nonzero mean. Case IV is the same except that the phasors of both channels are allowed to have specified nonzero means.

Tullsson [2] has pointed out that it may not be reasonable to neglect the noise correlation between the sum and difference channels and considered a correlation between the real parts and between the imaginary parts of the two channels. Case V treats the simple case where all the mean values are zero, while Case VI considers general means. Finally, Case VII considers an arbitrary fourth-order covariance matrix as well as arbitrary means.

It is well known that the density function of the monopulse ratio has a mean value but infinite variance. That is, the integral defining the variance diverges because of the long tails of the density function. Of course, in any practical situation the monopulse ratio would be bounded, and the variance could be computed. The essential consideration is the manner in which the density function should be modified to incorporate such limitations on the permissible values of the monopulse ratio in order to arrive at a satisfactory value for the variance. Tullsson [2] suggests a "threshholding" idea, according to which measurements having a sum-channel magnitude smaller than a given quantity are discarded. This is equivalent to considering a modified Gaussian density where a circular region in the (r, s) plane is excluded. The resulting monopulse-ratio densities are difficult to calculate in general, and are presented only in a few simple cases.

Let $\bar{x}, \bar{y}, \bar{u}, \bar{v}$ be the mean values of x, y, u, v, and let $\sigma_{xy}, \sigma_{xu}...$ be the components of the covariance matrix. "Isotropy", as in Cases I through VI in Chapters 4 and 5, is understood to mean that there are the relations

$$\sigma_{xx} = \sigma_{yy} = \sigma_x^2, \qquad \sigma_{uu} = \sigma_{vv} = \sigma_u^2 \tag{2.4}$$

defining the quantities σ_x and σ_u. "Uncorrelated", as in Cases I through IV of Chapter 4, is understood to mean that

$$\sigma_{xy} = \sigma_{xu} = \sigma_{xv} = \sigma_{yu} = \sigma_{yv} = \sigma_{uv} = 0 \tag{2.5}$$

CHAPTER 3

NUMERICAL GENERATION OF RANDOM VECTORS

It is useful to generate series of random vectors or linear arrays having a specified joint probability density. The density function of interest here is the joint Gaussian distribution of the four variables, x, y, u, v. However, the method to be discussed works for any number n of dimensions. First, obtain n independent normal variates, $\alpha_i, \sim N(a_i, 1), i = 1, ..., n$, and construct the desired n-variate by a linear transformation

$$X = QA \tag{3.1}$$

where $X = (x_1, x_2, ..., x_n)^T$ is the desired n-variate, $A = (\alpha_1, \alpha_2, ..., \alpha_n)^T$, and Q is the n by n transformation matrix to be determined. The α variables thus have specified means, unit variance, and are independently Gaussian distributed. The desired covariance matrix of X is

$$\Sigma = E[(X - \bar{X})(X - \bar{X})^T] = E[XX^T] - \bar{X}\bar{X}^T = QE[AA^T]Q^T - \bar{X}\bar{X}^T \tag{3.2}$$

where the overbar indicates the mean or expected value. According to the stated statistical properties of the α variables, it is seen that

$$E[AA^T] = \bar{A}\bar{A}^T + I \tag{3.3}$$

where I is the n by n identity matrix. Equation (3.2) then becomes

$$\begin{aligned} \Sigma &= Q(\bar{A}\bar{A}^T + I)Q^T - \bar{X}\bar{X}^T = Q\bar{A}\bar{A}^TQ^T + QQ^T - \bar{X}\bar{X}^T \\ &= \bar{X}\bar{X}^T + QQ^T - \bar{X}\bar{X}^T = QQ^T \end{aligned} \tag{3.4}$$

Since Σ is symmetric, a symmetric solution Q of (3.4) usually can be obtained by just taking the square root of the matrix Σ. There is usually a multiplicity of solutions, from which any one real square root matrix will provide the desired result. The means of the α variables are given by

$$\bar{A} = Q^{-1}\bar{X} \tag{3.5}$$

Some numerical computing packages (e.g., Matlab) have a matrix square root routine, which in many well-behaved cases determines a real square root of a real square matrix. If such a routine is not available, an eigenvalue routine can be used as follows. While there may be more elegant or more general procedures, the following meets the immediate need and is simple. Let λ and v be an eigenvalue and eigenvector of Q. Then,

$$Qv = \lambda v \tag{3.6}$$

Multiplying on the left by Q gives

$$QQv = \Sigma v = \lambda Qv = \lambda^2 v \tag{3.7}$$

showing that the eigenvalues of Q are just the square roots of the eigenvalues of Σ, and their eigenvectors are the same. Let V be the n by n matrix whose columns are the eigenvectors of Σ, and let L^2 be the diagonal matrix whose elements are the eigenvalues λ^2 of Σ. The matrix L is diagonal and has elements λ which are the square roots of the elements of L^2. Then

$$\begin{aligned} \Sigma V &= VL^2 \\ QV &= VL \end{aligned} \tag{3.8}$$

from which

$$Q = VLV^{-1} \tag{3.9}$$

is the desired solution.

These results are used to generate Monte Carlo stochastic samples of the (x, y, u, v) space having the desired density function, from which the resulting monopulse-ratio quantities (p, q) are obtained from the relations (3.2) and plotted as a scatter diagram. Such diagrams provide a useful demonstration of the validity of the density expressions obtained by analysis. The method is also useful for empirical determination of density functions in cases where analytic solutions are not available. It will be noted in subsequent chapters that the marginal density function for p alone (or q alone) is often difficult to obtain analytically, in which case the Monte Carlo method can provide a useful estimate. However, analytic solutions are preferable because their dependence on the various parameters is immediately visible.

CHAPTER 4

UNCORRELATED ISOTROPIC NOISE

The simplest although not the most realistic case is where the noise covariance matrix is diagonal. In accordance with the assumed isotropy the covariance noise matrix is assumed to be

$$\Sigma = \begin{pmatrix} \sigma_x^2 & 0 & 0 & 0 \\ 0 & \sigma_x^2 & 0 & 0 \\ 0 & 0 & \sigma_u^2 & 0 \\ 0 & 0 & 0 & \sigma_u^2 \end{pmatrix} \tag{4.1}$$

where σ_y is taken equal to σ_x, and σ_v equal to σ_u. Four subcases are considered by first taking the means of all the primary variables (x, y, u, v) to be zero amd finally allowing these means to have arbitrary values.

CASE I – Zero Means

The means $\bar{x}, \bar{y}, \bar{u}, \bar{v}$ are taken to be zero. The density distribution of the variables x, y, u, v is accordingly given by

$$P_{x,y,u,v}(x, y, u, v) = \frac{1}{4\pi^2\sigma_x^2\sigma_u^2} \exp\left[-\frac{x^2+y^2}{2\sigma_x^2} - \frac{u^2+v^2}{2\sigma_u^2}\right] \tag{4.2}$$

According to the statistical theorem stated in Chapter 2, the density function of the variables p, q, r, s is given by

$$\begin{aligned} P_{p,q,r,s}(p,q,r,s) &= P_{x,y,u,v}(pr - qs, ps + qr, r, s) \left|\frac{\partial(x,y,u,v)}{\partial(p,q,r,s)}\right| \\ &= \frac{r^2+s^2}{4\pi^2\sigma_x^2\sigma_u^2} \exp\left[-\frac{(pr-qs)^2+(ps+qr)^2}{2\sigma_x^2} - \frac{r^2+s^2}{2\sigma_u^2}\right] \\ &= \frac{r^2+s^2}{4\pi^2\sigma_x^2\sigma_u^2} \exp\left\{-(r^2+s^2)\left[\left(\frac{p^2+q^2}{2\sigma_x^2}\right) + \frac{1}{2\sigma_u^2}\right]\right\} \end{aligned} \tag{4.3}$$

The marginal density of the p, q variables is found by integrating the above expression over the entire (r, s) plane. The substitution

$$\begin{aligned} r &= R\cos\theta \\ s &= R\sin\theta \end{aligned} \tag{4.4}$$

gives

$$P_{p,q}(p,q) = \frac{1}{4\pi^2\sigma_x^2\sigma_u^2} \int_0^\infty R^3 dR \int_0^{2\pi} \exp\left[-R^2\left(\frac{p^2+q^2}{2\sigma_x^2} + \frac{1}{2\sigma_u^2}\right)\right] d\theta \tag{4.5}$$

The result is

$$P_{p,q}(p,q) = \frac{\gamma^2}{\pi[p^2+q^2+\gamma^2]^2} \tag{4.6}$$

where γ is written for the ratio σ_x/σ_u of the standard deviations (see Appendix for details). This result has been presented by Tullsson [2] under somewhat more general conditions. It is noteworthy that the above joint density distribution depends not on the two individual variances but only on their ratio.

The density function (4.6) can be expressed in more compact form containing no parameter by defining

$$p' = p/\gamma, \quad q' = q/\gamma, \quad P'_{p,q} = \gamma^2 P_{p,q} \tag{4.7}$$

Then

$$P'_{p,q}(p',q') = \frac{1}{\pi(p'^2+q'^2+1)^2} \tag{4.8}$$

Essentially the parameter γ has disappeared through appropriate scaling. This function is shown graphically in Figures 4-1 and 4-2 at the end of this chapter.

It is easy to find the marginal density of p by integrating (4.6) with respect to q. We have

$$\begin{aligned} P_p(p) &= \frac{\gamma^2}{\pi}\int_{-\infty}^{\infty}\frac{dq}{[p^2+q^2+\gamma^2]^2} \\ &= \tfrac{1}{2}\gamma^2(p^2+\gamma^2)^{-3/2} \end{aligned} \tag{4.9}$$

A similar compact form of the marginal density (4.9) is obtained through the scaling (4.7) along with the definition

$$P'_p = \gamma P_p \tag{4.10}$$

giving

$$P'_p(p') = \frac{1}{\pi(p'^2+1)^{3/2}} \tag{4.11}$$

This function is shown in Figure 4-3. There are similar expressions for the marginal density of q. It is readily demonstrated that the integral of $P_p(p)$ in (4.9) over the entire p axis is unity. The mean value, $\bar{p}$, is zero because (4.9) is an even function of p while the integral defining $\bar{p}$ converges. However, the variance of p is indeterminate as it is expressed by a divergent integral.

CASE II – Zero Mean of the Difference Channel Phasor

The covariance matrix (4.1) and means $\bar{x}$ and $\bar{y}$ of zero are assumed, while $\bar{u}$ and $\bar{v}$ can be assigned. The density function for the primary variables is

$$P_{x,y,u,v}(x,y,u,v) = \frac{1}{4\pi^2\sigma_x^2\sigma_u^2}\exp\left[-\frac{x^2+y^2}{2\sigma_x^2} - \frac{(u-\bar{u})^2+(v-\bar{v})^2}{2\sigma_u^2}\right] \tag{4.12}$$

The density function of the secondary variables is then given by

$$
\begin{aligned}
P_{p,q,r,s}(p,q,r,s) &= \frac{r^2+s^2}{4\pi^2\sigma_x^2\sigma_u^2}\exp\left[-\frac{(pr-qs)^2+(ps+qr)^2}{2\sigma_x^2}-\frac{(r-\bar{u})^2+(s-\bar{v})^2}{2\sigma_u^2}\right] \\
&= \frac{r^2+s^2}{4\pi^2\sigma_x^2\sigma_u^2}\exp\left[-U-\left(\frac{p^2+q^2}{2\sigma_x^2}+\frac{1}{2\sigma_u^2}\right)(r^2+s^2)+\frac{\bar{u}r+\bar{v}s}{\sigma_u^2}\right]
\end{aligned} \tag{4.13}
$$

where

$$
U=\frac{\bar{u}^2+\bar{v}^2}{2\sigma_u^2} \tag{4.14}
$$

To find the marginal distribution of (p,q) the substitution (4.4) is again used to obtain

$$
\begin{aligned}
P_{p,q}(p,q) = \frac{e^{-U}}{4\pi^2\sigma_x^2\sigma_u^2}\int_0^\infty R^3 dR\,\exp\left[-\left(\frac{p^2+q^2}{2\sigma_x^2}+\frac{1}{2\sigma_u^2}\right)R^2\right]\times \\
\int_0^{2\pi}\exp\left[\frac{R(\bar{u}\cos\theta+\bar{v}\sin\theta)}{\sigma_u^2}\right]d\theta
\end{aligned} \tag{4.15}
$$

The details of the integrations are given in the Appendix. The inner integral is

$$
\int_0^{2\pi}\exp\left[\frac{R(\bar{u}\cos\theta+\bar{v}\sin\theta)}{\sigma_u^2}\right]d\theta = 2\pi I_0\left(\frac{R}{\sigma_u^2}\sqrt{\bar{u}^2+\bar{v}^2}\right) \tag{4.16}
$$

where I_0 is the Modified Bessel Function of zero order. The outer integral is then

$$
P_{p,q}(p,q)=\frac{e^{-U}}{2\pi\sigma_x^2\sigma_u^2}\int_0^\infty R^3 dR\,\exp\left[-\left(\frac{p^2+q^2}{2\sigma_x^2}+\frac{1}{2\sigma_u^2}\right)R^2\right]I_0\left(\frac{R}{\sigma_x^2}\sqrt{\bar{u}^2+\bar{v}^2}\right) \tag{4.17}
$$

which becomes

$$
P_{p,q}(p,q)=\frac{\mu^2(1+\mu)e^{\mu-U}}{\pi U^2\gamma^2} \tag{4.18}
$$

where μ stands for the quantity

$$
\mu=\frac{U\gamma^2}{p^2+q^2+\gamma^2}=\frac{U}{p'^2+q'^2+1} \tag{4.19}
$$

It is reasily demonstrated (see Appendix) that the total volume under the joint density function (4.18) is unity, and it is seen that (4.18) reduces to (4.6) of Case I if $U\to 0$. The compact form of (4.18) is

$$
P'_{p,q}(p',q')=\frac{\mu^2(1+\mu)e^{\mu-U}}{\pi U^2} \tag{4.20}
$$

with the definitions (4.7) and (4.19). This function is shown in Figure 4-4 for various values of U.

The marginal density of p is

$$
P_p(p)=\frac{e^{-U}}{\pi U^2\gamma^2}\int_{-\infty}^{\infty}\mu^2(1+\mu)e^{\mu}dq \tag{4.21}
$$

As μ depends on q it is convenient to use μ as the variable of integration. The details are given in the Appendix. After the transformation the density function is

$$P_p(p) = \frac{e^{-U}}{\pi U} \int_0^{U\gamma^2/(p^2+\gamma^2)} \frac{\mu(1+\mu)e^{\mu} d\mu}{\sqrt{U\gamma^2\mu - (p^2+\gamma^2)\mu^2}} \tag{4.22}$$

which finally leads to

$$P_p(p) = \frac{\gamma^2 e^{-U}}{2(p^2+\gamma^2)^{3/2}} \sum_{j=0}^{\infty} \frac{(2j+1)!}{[j!]^3} \left(\frac{U\gamma^2}{4(p^2+\gamma^2)}\right)^j \tag{4.23}$$

or to

$$P'_p(p') = \frac{e^{-U}}{2(p'^2+1)^{3/2}} \sum_{j=0}^{\infty} \frac{(2j+1)!}{[j!]^3} \left(\frac{U}{4(p'^2+1)}\right)^j \tag{4.24}$$

In terms of standard functions these expressions are equivalent to

$$\begin{aligned} P_p(p) = {} & \frac{\gamma^2}{2(p^2+\gamma^2)^{3/2}} \exp\left[-\frac{U}{2}\left(\frac{2p^2+\gamma^2}{p^2+\gamma^2}\right)\right] \times \\ & \left[\left(1+\frac{U\gamma^2}{p^2+\gamma^2}\right) I_0\left(-\frac{U\gamma^2}{2(p^2+\gamma^2)}\right) - \frac{U\gamma^2}{p^2+\gamma^2} I_1\left(-\frac{U\gamma^2}{2(p^2+\gamma^2)}\right)\right] \end{aligned} \tag{4.25}$$

or to

$$\begin{aligned} P'_p(p') = {} & \frac{1}{2(p'^2+1)^{3/2}} \exp\left[-\frac{U}{2}\left(\frac{2p'^2+1}{p'^2+1}\right)\right] \times \\ & \left[\left(1+\frac{U}{p'^2+1}\right) I_0\left(-\frac{U}{2(p'^2+1)}\right) - \frac{U}{p'^2+1} I_1\left(-\frac{U}{2(p'^2+1)}\right)\right] \end{aligned} \tag{4.26}$$

where I_0 and I_1 are Modified Bessel functions. This density function is shown in Figure 4-5 for various values of U.

Again it is noted that $\bar{p} = 0$ and the variance of p is infinite. The expressions (4.25) and (4.26) reduce to (4.9) of Case I when $U \to 0$ because $I_0(0) = 1$ and $I_1(0) = 0$.

CASE III – Zero Mean of the Sum Channel Phasor

The covariance matrix (4.1) and means $\bar{u}$ and $\bar{v}$ of zero are assumed, while $\bar{x}$ and $\bar{y}$ can be assigned. For this case the density function for the primary variables is

$$P_{x,y,u,v}(x,y,u,v) = \frac{1}{4\pi^2\sigma_x^2\sigma_u^2} \exp\left[-\frac{(x-\bar{x})^2+(y-\bar{y})^2}{2\sigma_x^2} - \frac{u^2+v^2}{2\sigma_u^2}\right] \tag{4.27}$$

The density function of the secondary variables is then given by

$$\begin{aligned} P_{p,q,r,s}(p,q,r,s) &= \frac{r^2+s^2}{4\pi^2\sigma_x^2\sigma_u^2} \exp\left[-\frac{(pr-qs-\bar{x})^2+(ps+qr-\bar{y})^2}{2\sigma_x^2} - \frac{r^2+s^2}{2\sigma_u^2}\right] \\ &= \frac{r^2+s^2}{4\pi^2\sigma_x^2\sigma_u^2} \exp\left[-X - \frac{(r^2+s^2)(p^2+q^2) - 2p(r\bar{x}+s\bar{y}) + 2q(s\bar{x}-r\bar{y})}{2\sigma_x^2} - \frac{r^2+s^2}{2\sigma_u^2}\right] \end{aligned} \tag{4.28}$$

where X stands for the quantity

$$X = \frac{\bar{x}^2 + \bar{y}^2}{2\sigma_x^2} \tag{4.29}$$

The marginal distribution of p and q is again found by means of the substitution (4.4), to obtain

$$\begin{aligned} P_{p,q}(p,q) = & \frac{e^{-X}}{4\pi^2\sigma_x^2\sigma_u^2}\int_0^\infty R^3 dR \exp\left[-R^2\left(\frac{p^2+q^2}{2\sigma_x^2}+\frac{1}{2\sigma_u^2}\right)\right] \times \\ & \int_0^{2\pi} d\theta \exp\left[\frac{R}{\sigma_x^2}([p\bar{x}+q\bar{y}]\cos\theta + [p\bar{y}-q\bar{x}]\sin\theta)\right] \end{aligned} \tag{4.30}$$

The inner integral is evaluated in same way (4.17) was obtained from (4.15). The result is

$$P_{p,q}(p,q) = \frac{e^{-X}}{2\pi\sigma_x^2\sigma_u^2}\int_0^\infty R^3 dR \exp\left[-R^2\left(\frac{p^2+q^2}{2\sigma_x^2}+\frac{1}{2\sigma_u^2}\right)\right] I_0\left[\frac{R}{\sigma_x^2}\sqrt{(p^2+q^2)(\bar{x}^2+\bar{y}^2)}\right] \tag{4.31}$$

This integration is performed in the same way that (4.18) was obtained from (4.17), giving

$$P_{p,q}(p,q) = \frac{\gamma^2\mu^2(1+\mu)e^{\mu-X}}{\pi X^2(p^2+q^2)^2} \tag{4.32}$$

or

$$P'_{p,q}(p',q') = \frac{\mu^2(1+\mu)e^{\mu-X}}{\pi X^2(p'^2+q'^2)^2} \tag{4.33}$$

where μ is now defined

$$\mu = \frac{(p^2+q^2)X}{p^2+q^2+\gamma^2} = \frac{(p'^2+q'^2)X}{p'^2+q'^2+1} \tag{4.34}$$

Again, $\bar{p} = 0$ because (4.32) is an even function of p and the integral defining $\bar{p}$ converges. The integral defining the variance does not converge, so again no finite variance exists. The expression (4.32) reduces to (4.6) of Case I when $X \to 0$. The marginal distribution of p is somewhat more cumbersome to obtain than in the previous cases, and has not been developed. Figure 4-6 shows this joint density function for various values of X. It is noted that there is a minimum, or depression, at the origin for values of X greater than approximately unity.

CASE IV – General Means

For this case the covariance matrix (4.1) is assumed while the means of all four primary variables x, y, u, v are specified to be $\bar{x}, \bar{y}, \bar{u}, \bar{v}$. The density function of the primary variables is accordingly given by

$$P_{x,y,u,v}(x,y,u,v) = \frac{1}{4\pi^2\sigma_x^2\sigma_y^2}\exp\left[-\frac{(x-\bar{x})^2+(y-\bar{y})^2)}{2\sigma_x^2} - \frac{(u-\bar{u})^2+(v-\bar{v})^2}{2\sigma_u^2}\right] \tag{4.35}$$

The corresponding density function for the secondary variables is

$$P_{p,q,r,s}(p,q,r,s) = \frac{r^2+s^2}{4\pi^2\sigma_x^2\sigma_u^2} \exp\left[-\frac{(pr-qs-\bar{x})^2+(ps+qr-\bar{y})^2}{2\sigma_x^2} - \frac{(r-\bar{u})^2+(s-\bar{v})^2}{2\sigma_u^2}\right]$$
$$= \frac{r^2+s^2}{4\pi^2\sigma_x^2\sigma_u^2} \exp\left[-\left(\frac{p^2+q^2}{2\sigma_x^2}+\frac{1}{2\sigma_u^2}\right)(r^2+s^2) - X - U\right] \times$$
$$\exp\left[\left(\frac{\bar{x}p+\bar{y}q}{\sigma_x^2}+\frac{\bar{u}}{\sigma_u^2}\right)r + \left(\frac{\bar{y}p-\bar{x}q}{\sigma_x^2}+\frac{\bar{v}}{\sigma_u^2}\right)s\right] \tag{4.36}$$

The integration can be carried out in the manner that was used to obtain (4.32) from (4.28). The result is

$$P_{p,q}(p,q) = \frac{\gamma^2(1+\mu)\,e^{\mu-X-U}}{\pi(p^2+q^2+\gamma^2)^2} \tag{4.37}$$

or

$$P'_{p,q}(p',q') = \frac{(1+\mu)e^{\mu-X-U}}{\pi(p'^2+q'^2+1)^2} \tag{4.38}$$

where here μ is redefined to be

$$\mu = \frac{X(p^2+q^2)+\gamma(Wp+Zq)+\gamma^2 U}{p^2+q^2+\gamma^2} = \frac{X(p'^2+q'^2)+Wp'+Zq'+U}{p'^2+q'^2+1} \tag{4.39}$$

with the parameters

$$W = \frac{\bar{u}\bar{x}+\bar{v}\bar{y}}{\sigma_x\sigma_u}, \qquad Z = \frac{\bar{u}\bar{y}-\bar{v}\bar{x}}{\sigma_x\sigma_u} \tag{4.40}$$

which depend only on the density function of the primary variables, being defined for convenience. When $\bar{x}=\bar{y}=0$, (4.37) reduces to (4.18) of case II. When $\bar{u}=\bar{v}=0$, (4.37) reduces to (4.32) of Case III. The density (4.37) is not an even function because of the presence of p and q to the first power in the expression (4.39) for μ. Consequently $\bar{p}$ and $\bar{q}$ do not vanish as they did in previous cases. Figures 4-7 and 4-8 show the function (4.37) for the case

$$\bar{x}=2, \quad \bar{y}=2, \quad \bar{u}=5, \quad \bar{v}=0, \quad \sigma_x=1, \quad \sigma_u=20 \tag{4.41}$$

Figure 4.9 shows various perspectives of the same function. A Monte Carlo simulation was carried out by selecting random values of the primary variables from the density function defined by (4.35) and calculating the corresponding complex monopulse ratios. These are plotted on the complex plane as a scatter diagram and shown in Figure 4-10 as an experimental validation of the derived joint density function (4.37). Thus, Figure 4-9 is an empirical counterpart to the joint density function depicted in Figures 4-7, 4-8, and 4-9.

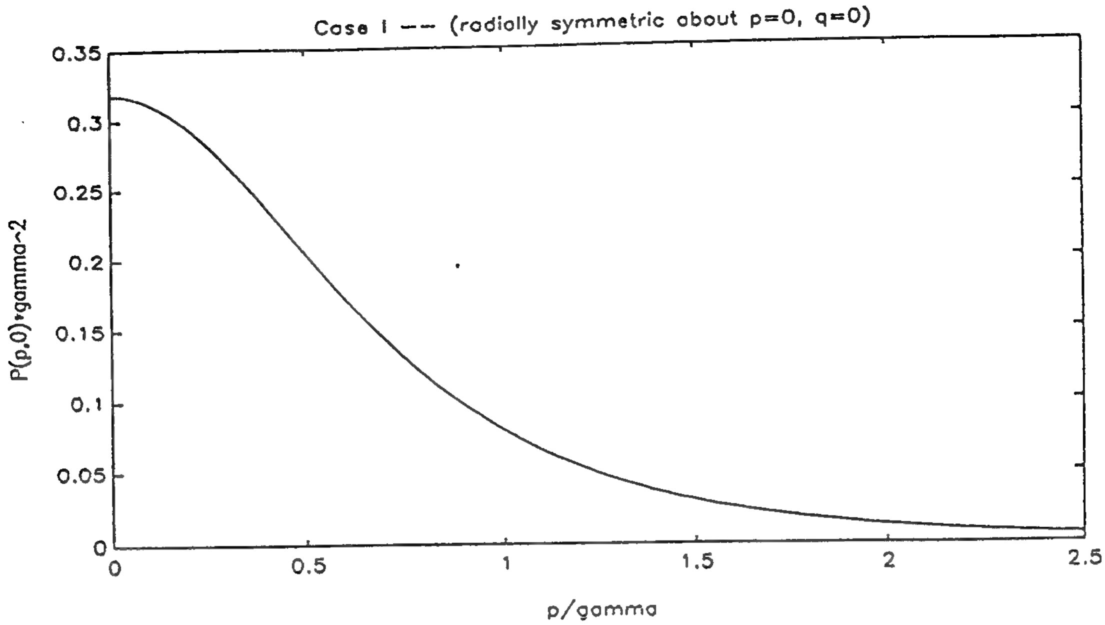

FIGURE 4-1. JOINT MR DENSITY, UNCORRELATED NOISE, ZERO MEANS

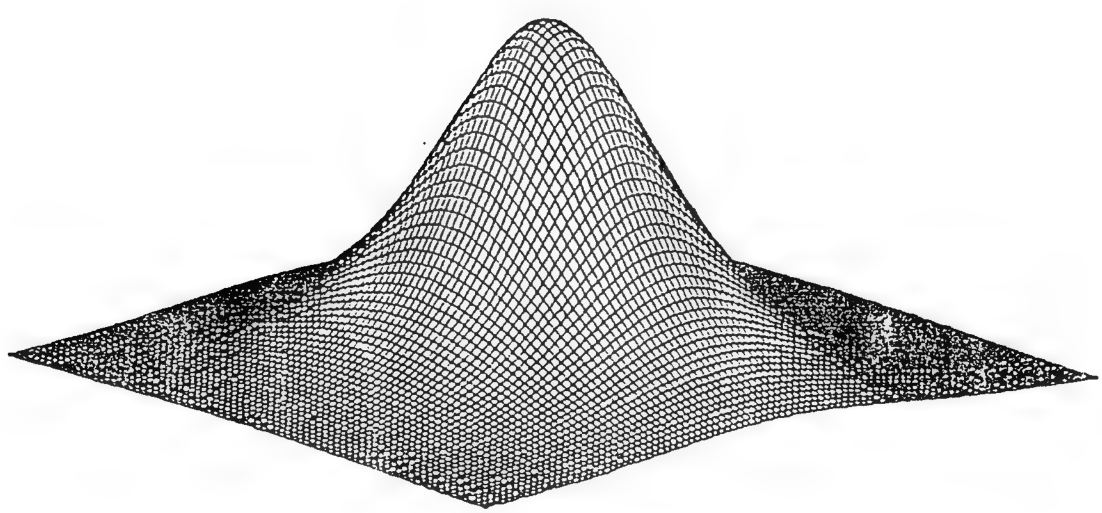

FIGURE 4-2. JOINT MR DENSITY, UNCORRELATED NOISE, ZERO MEANS

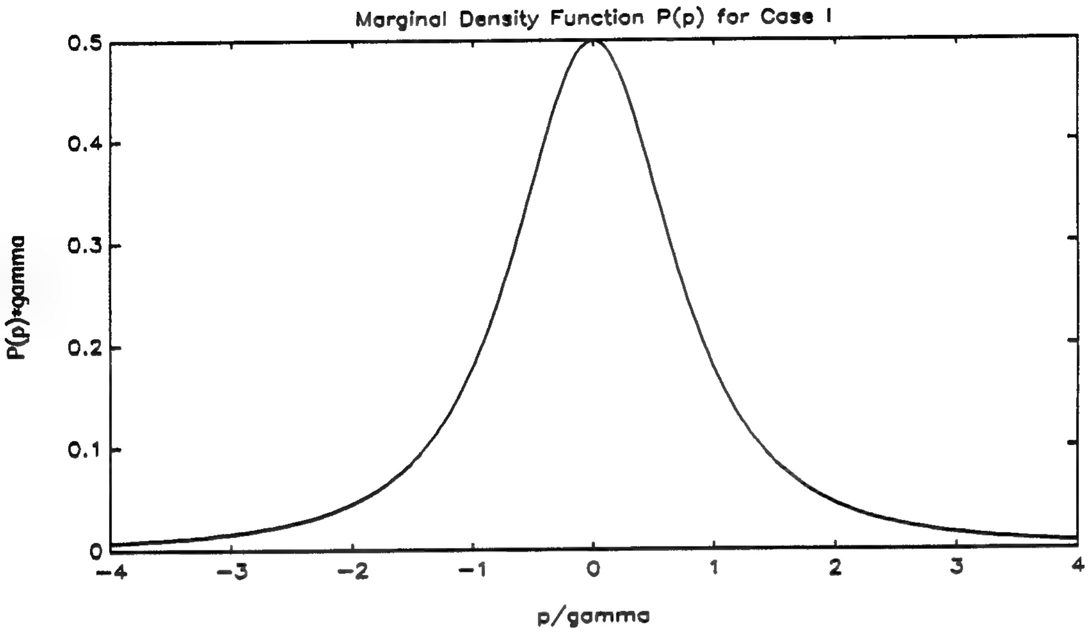

FIGURE 4-3. REAL MR DENSITY, UNCORRELATED NOISE, ZERO MEANS

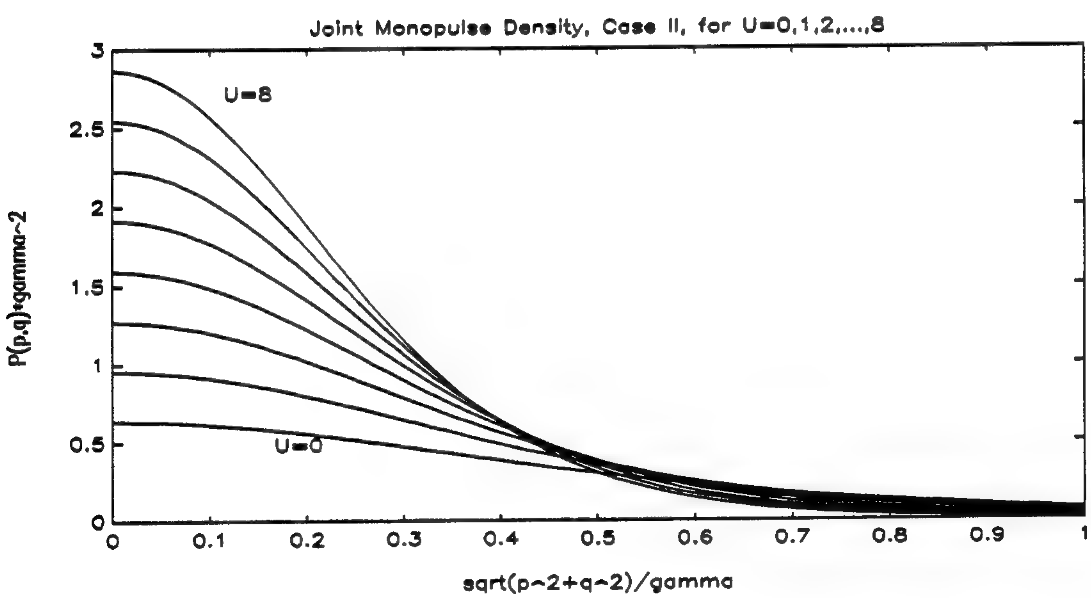

FIGURE 4-4. JOINT MR DENSITY, UNCORRELATED NOISE, ZERO DIFFERENCE MEAN

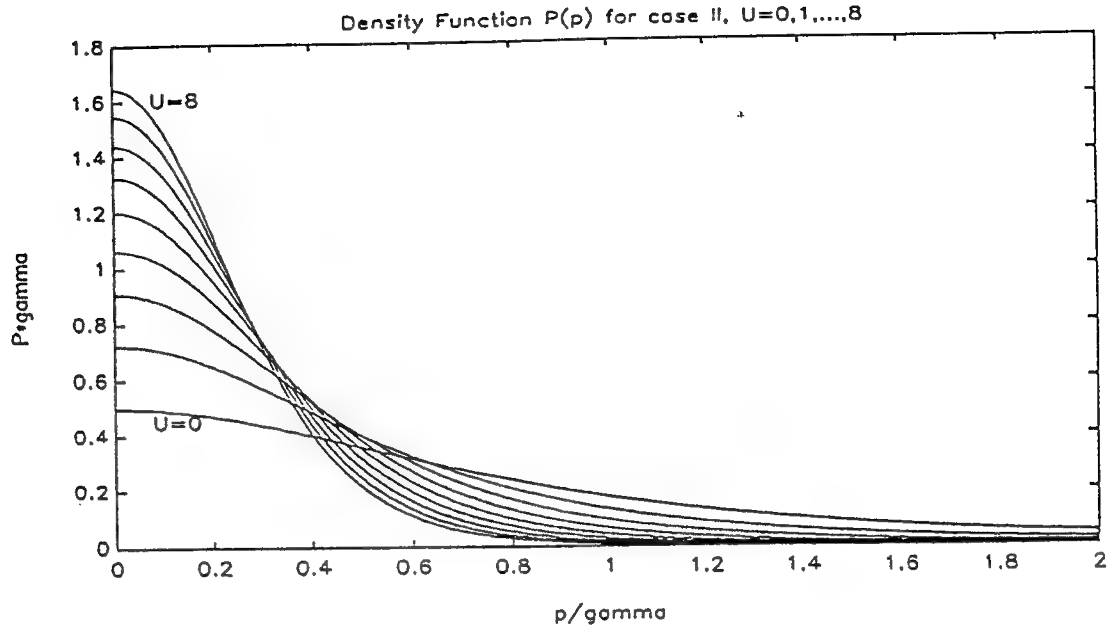

FIGURE 4-5. REAL MR DENSITY, UNCORRELATED NOISE, ZERO DIFFERENCE MEAN

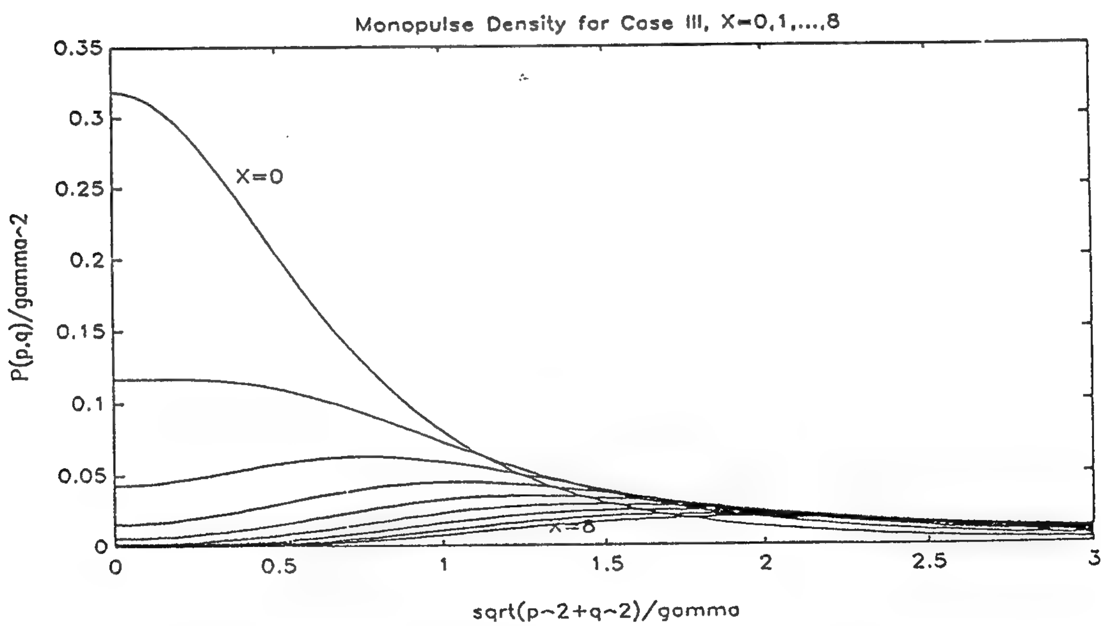

FIGURE 4-6. JOINT MR DENSITY, UNCORRELATED NOISE, ZERO SUM MEAN

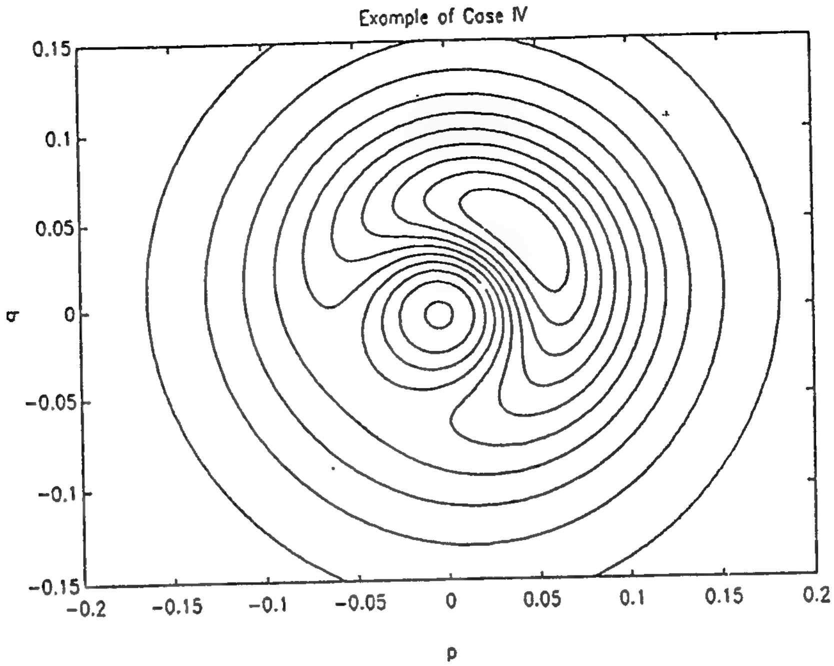

FIGURE 4-7. JOINT MR DENSITY, UNCORRELATED NOISE, GENERAL MEANS

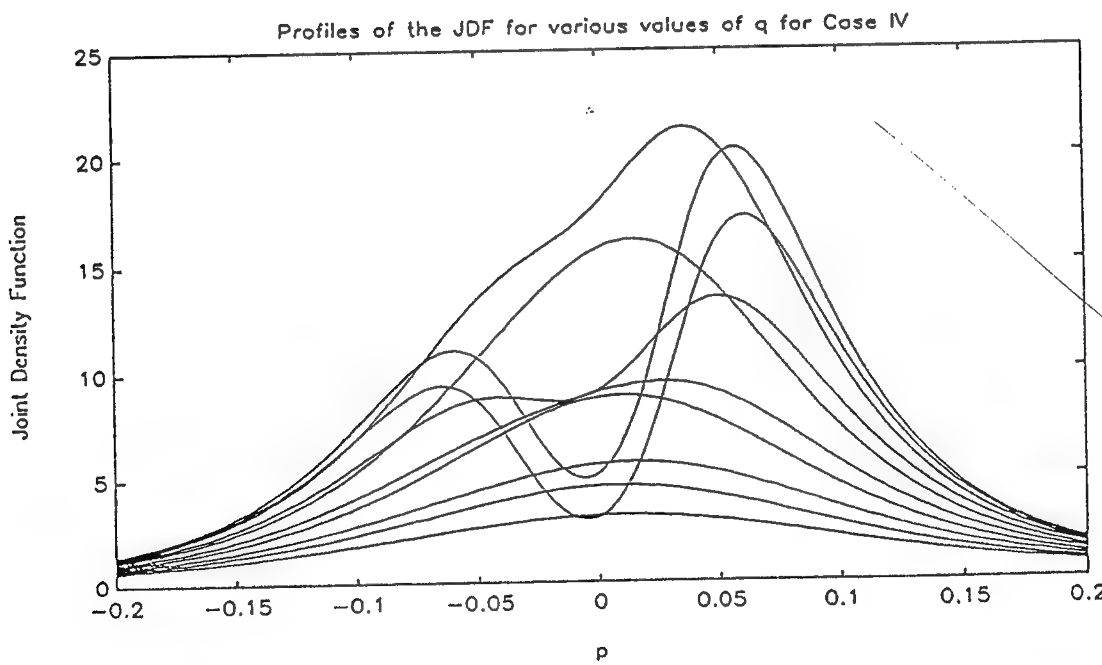

FIGURE 4-8. JOINT MR DENSITY, UNCORRELATED NOISE, GENERAL MEANS

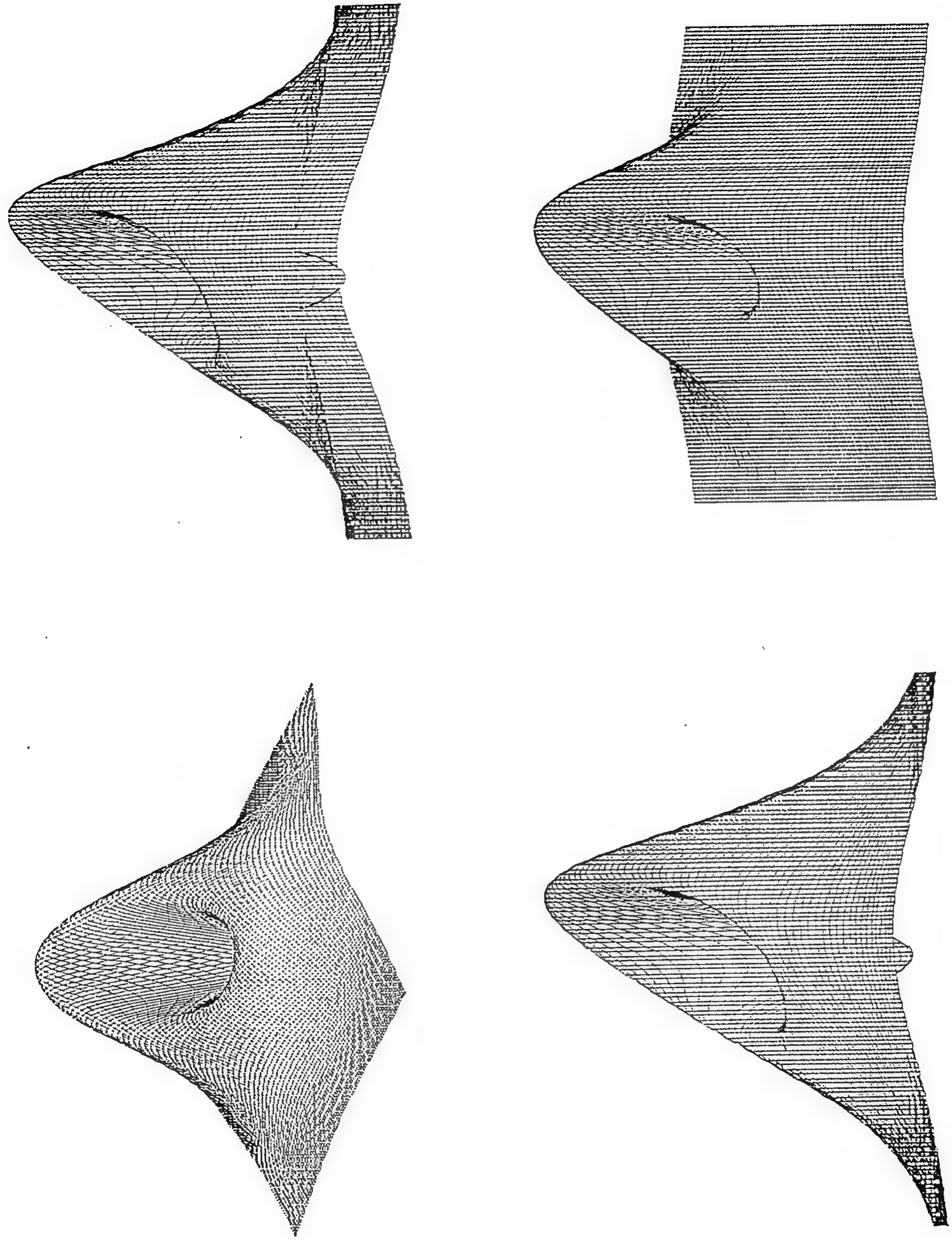

FIGURE 4-9. JOINT MR DENSITY, UNCORRELATED NOISE, GENERAL MEANS

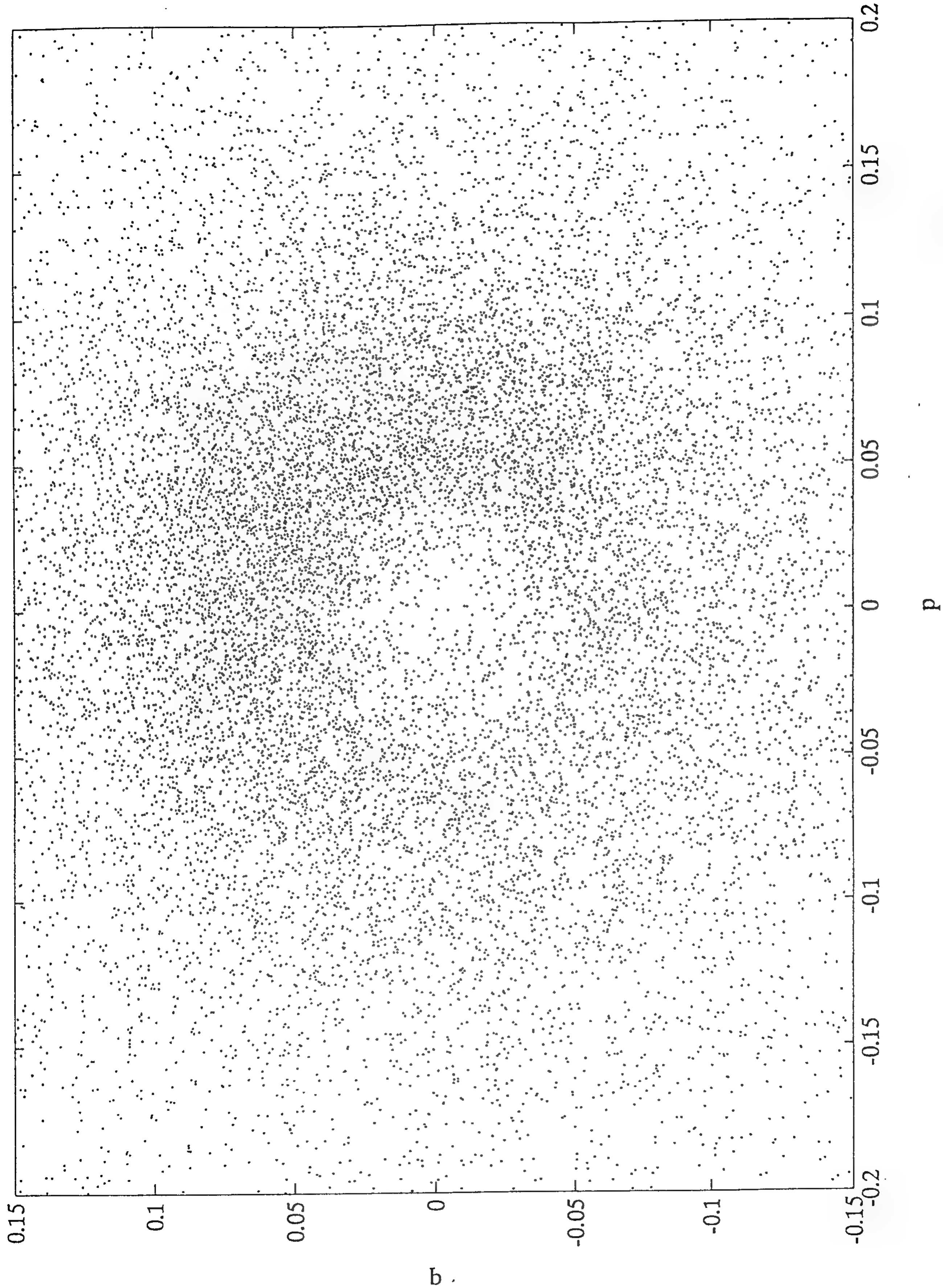

FIGURE 4-10. MONTE CARLO SCATTER DIAGRAM OF MR FOR UNCORRELATED NOISE

CHAPTER 5

PARTIALLY CORRELATED ISOTROPIC NOISE

Noise effects in the sum and difference phasors are correlated because they are linear combinations of the same antenna lobe outputs. We shall consider the noise covariance matrix suggested by Tullsson [2], according to whom the assumption of real antenna patterns gives a correlation between the real parts and between the imaginary parts but not crosswise. Accordingly, the covariance matrix can be represented

$$\Sigma = \begin{pmatrix} \sigma_x^2 & 0 & \rho\sigma_x\sigma_u & 0 \\ 0 & \sigma_x^2 & 0 & \rho\sigma_x\sigma_u \\ \rho\sigma_x\sigma_u & 0 & \sigma_u^2 & 0 \\ 0 & \rho\sigma_x\sigma_u & 0 & \sigma_u^2 \end{pmatrix} = \sigma_u^2 \begin{pmatrix} \gamma^2 & 0 & \rho\gamma & 0 \\ 0 & \gamma^2 & 0 & \rho\gamma \\ \rho\gamma & 0 & 1 & 0 \\ 0 & \rho\gamma & 0 & 1 \end{pmatrix} \tag{5.1}$$

where ρ is a correlation coefficient. The determinant and inverse of (5.1) are

$$|\Sigma| = \sigma_x^4\sigma_u^4(1-\rho^2)^2, \qquad \Sigma^{-1} = \frac{1}{\sigma_x^2(1-\rho^2)} \begin{pmatrix} 1 & 0 & -\rho\gamma & 0 \\ 0 & 1 & 0 & -\rho\gamma \\ -\rho\gamma & 0 & \gamma^2 & 0 \\ 0 & -\rho\gamma & 0 & \gamma^2 \end{pmatrix} \tag{5.2}$$

CASE V – Zero Means

When the means $\bar{x}, \bar{y}, \bar{u}, \bar{v}$ are taken to be zero, the density function of the primary variables is

$$P_{x,y,u,v}(x,y,u,v) = \frac{1}{4\pi^2\sigma_x^2\sigma_u^2(1-\rho^2)} \times \exp\left[-\frac{\sigma_u^2x^2 - 2\rho\sigma_x\sigma_u xu + \sigma_x^2u^2 + \sigma_u^2y^2 - 2\rho\sigma_x\sigma_u yv + \sigma_x^2v^2}{2\sigma_x^2\sigma_u^2(1-\rho^2)}\right] \tag{5.3}$$

The corresponding density function of the secondary variables is then

$$P_{p,q,r,s}(p,q,r,s) = \frac{r^2+s^2}{4\pi^2\sigma_x^2\sigma_u^2(1-\rho^2)} \times \exp\left[-\frac{\sigma_u^2(pr-qs)^2 - \rho\sigma_x\sigma_u r(pr-qs) + \sigma_x^2r^2 + \sigma_u^2(ps+qr)^2 - 2\rho\sigma_x\sigma_u s(ps+qr) + \sigma_x^2s^2}{2\sigma_x^2\sigma_u^2(1-\rho^2)}\right] \tag{5.4}$$

After some rearrangement this becomes

$$P_{p,q,r,s}(p,q,r,s) = \frac{r^2+s^2}{4\pi^2\sigma_x^2\sigma_u^2(1-\rho^2)} \exp\left[-\frac{(p^2+q^2-2\rho\gamma p+\gamma^2)(r^2+s^2)}{2\sigma_x^2(1-\rho^2)}\right] \tag{5.5}$$

The marginal distribution of the variables (p, q) is obtained as in (4.6) with the result

$$P_{p,q}(p,q) = \frac{\gamma^2(1-\rho^2)}{\pi(p^2+q^2-2\rho\gamma p+\gamma^2)^2} \tag{5.6}$$

This function reduces to the joint density (4.6) if ρ is set to zero. Equation (5.6) is illustrated in Figures 5-1 and 5-2 for various values of ρ. The compact form is

$$P'_{p,q}(p',q') = \frac{1-\rho^2}{\pi(p'^2+q'^2-2\rho p'+1)^2} \tag{5.7}$$

The marginal distributions are

$$P_p(p) = \int_{-\infty}^{\infty} P_{p,q}(p,q)dq = \frac{\gamma^2(1-\rho^2)}{2(p^2-2\rho\gamma p+\gamma^2)^{3/2}} \tag{5.8}$$

and

$$P_q(q) = \int_{-\infty}^{\infty} P_{p,q}(p,q)dp = \frac{\gamma^2(1-\rho^2)}{2[q^2+\gamma^2(1-\rho^2)]^{3/2}} \tag{5.9}$$

The corresponding compact forms are

$$P'_p(p') = \frac{(1-\rho^2)}{2(p'^2-2\rho p'+1)^{3/2}} \tag{5.10}$$

and

$$P'_q(q') = \frac{(1-\rho^2)}{2[q'^2+(1-\rho^2)]^{3/2}} \tag{5.11}$$

The density functions (5.10) and (5.11) are illustrated in Figures 5-3 and 5-4 for various values of ρ. It is seen that $\bar{q} = 0$. If the substitution $p = t + \gamma\rho$ is made in (5.8), it is seen that $\bar{t} = 0$, which shows that $\bar{p} = \rho\gamma$. The variances of both p and q diverge as in the case of uncorrelated noise.

CASE VI – General Means

When the means of the primary variables can be specified, the joint density function of the secondary variables is

$$P_{p,q,r,s}(p,q,r,s) = \frac{r^2+s^2}{4\pi^2\sigma_x^2\sigma_u^2(1-\rho^2)}\exp\left[-\frac{A(r^2+s^2)+Br+Cs+D}{2\sigma_x^2(1-\rho^2)}\right] \tag{5.12}$$

where A, B, C, and D, and their primed counterparts are defined by

$$\begin{aligned}
\gamma^2 A' &= A = (p^2+q^2) - 2\rho\gamma p + \gamma^2 \\
\gamma\sigma_x B' &= B = -2(\bar{x}p+\bar{y}q) + 2\rho\gamma(\bar{x}+\bar{u}p+\bar{v}q) - 2\gamma^2\bar{u} \\
\gamma\sigma_x C' &= C = 2(\bar{x}q-\bar{y}p) - 2\rho\gamma(\bar{u}q-\bar{y}-\bar{v}p) - 2\gamma^2\bar{v} \\
\sigma_x^2 D' &= D = (\bar{x}^2+\bar{y}^2) - 2\rho\gamma(\bar{v}\bar{y}+\bar{u}\bar{x}) + \gamma^2(\bar{u}^2+\bar{v}^2)
\end{aligned} \tag{5.13}$$

all being independent of r and s. A feature of (5.12) which facilitates further development is the absence of a product rs either inside or outside the exponential factor. To get the marginal bivariate density of p and q, the integration with respect to r and s is again carried out with the aid of the transformation (4.4) to obtain

$$P_{p,q}(p,q) = \frac{1}{4\pi^2\sigma_x^2\sigma_u^2(1-\rho^2)} \int_0^\infty R^3 \exp\left[-\frac{AR^2+D}{2\sigma_x^2(1-\rho^2)}\right] dR \times \int_0^{2\pi} \exp\left[-\frac{R(B\cos\theta + C\sin\theta)}{2\sigma_x^2(1-\rho^2)}\right] d\theta \tag{5.14}$$

The inner integration can be carried out in same way that (4.19) was obtained from (4.15). The result is

$$P_{p,q}(p,q) = \frac{1}{2\pi\sigma_x^2\sigma_u^2(1-\rho^2)} \int_0^\infty R^3 \exp\left[-\frac{AR^2+D}{2\sigma_x^2(1-\rho^2)}\right] I_0\left[\frac{R\sqrt{B^2+C^2}}{2\sigma_x^2(1-\rho^2)}\right] dR \tag{5.15}$$

The last integration can be carried out in the same way that (4.18) was obtained from (4.17), giving

$$P_{p,q}(p,q) = \frac{\gamma^2(1-\rho^2)}{\pi A^2}\left[1 + \frac{B^2+C^2}{8A\sigma_x^2(1-\rho^2)}\right] \exp\left(-\frac{4AD-B^2-C^2}{8A\sigma_x^2(1-\rho^2)}\right) \tag{5.16}$$

It can be shown that (5.16) reduces to (4.37) if $\rho \to 0$. An example of this joint density function is shown in Figures 5-5 and 5-6 for the values (4.41) and $\rho = .99$. Various perspectives of this function are shown in Figure 5-7. Comparing Figures 5-5, 5-6 and 5-7 for $\rho = .99$ with Figures 4-7, 4-8, and 4-9, for $\rho = 0$ reveals the effect of the correlation between the sum and difference phasors on the resulting joint MR density. In particular, it is noted that the correlated case has a deeper and flatter 'hole' or minimum. A Monte Carlo simulation was carried out in the manner of Figure 4-10 and the results are presented in Figure 5-8. Comparing the two scatter diagrams shows empirical confirmation that the correlation leads to a deeper minimum in the joint density.

A compact form of (5.16) is obtained by defining

$$\bar{x}' = \bar{x}/\sigma_x, \quad \bar{y}' = \bar{y}/\sigma_x, \quad \bar{u}' = \bar{u}/\sigma_u, \quad \bar{v}' = \bar{v}/\sigma_u \tag{5.17}$$

With these definitions the primed quantities defined in (5.13) become

$$\begin{aligned} A' &= p'^2 + q'^2 - 2\rho p' + 1 \\ B' &= -2(\bar{x}'p' + \bar{y}'q') + 2\rho(\bar{x}' + \bar{u}'p' + \bar{v}'q') - 2\bar{u}' \\ C' &= 2(\bar{x}'q' - \bar{y}'p') - 2\rho(\bar{u}'q' - \bar{y}' - \bar{v}'p') - 2\bar{v}' \\ D' &= \bar{x}'^2 + \bar{y}'^2 - 2\rho(\bar{v}'y' + \bar{u}'x') + \bar{u}'^2 + \bar{v}'^2 \end{aligned} \tag{5.18}$$

These, along with (4.7) allow (5.16) to be written in the form

$$P'_{p,q}(p,q) = \frac{1-\rho^2}{\pi A'^2}\left[1 + \frac{B'^2+C'^2}{8A'(1-\rho^2)}\right] \exp\left(-\frac{4A'D'-B'^2-C'^2}{8A'(1-\rho^2)}\right) \tag{5.19}$$

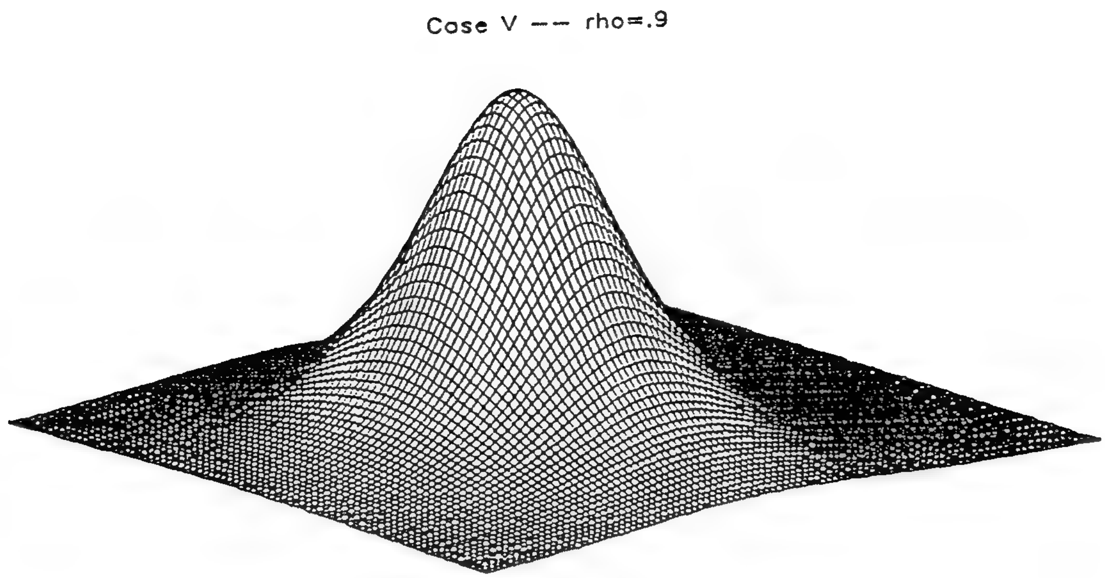

FIGURE 5-1. JOINT MR DENSITY, PARTIALLY CORRELATED NOISE

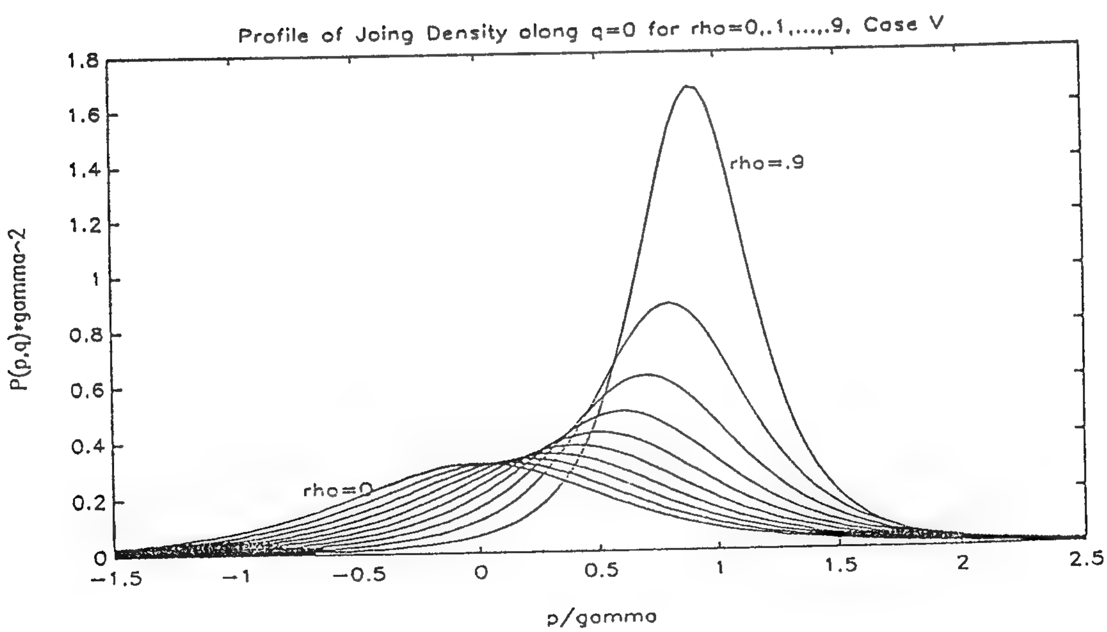

FIGURE 5-2. JOINT MR DENSITY FOR VARIOUS VALUES OF THE CORRELATION COEFFICIENT

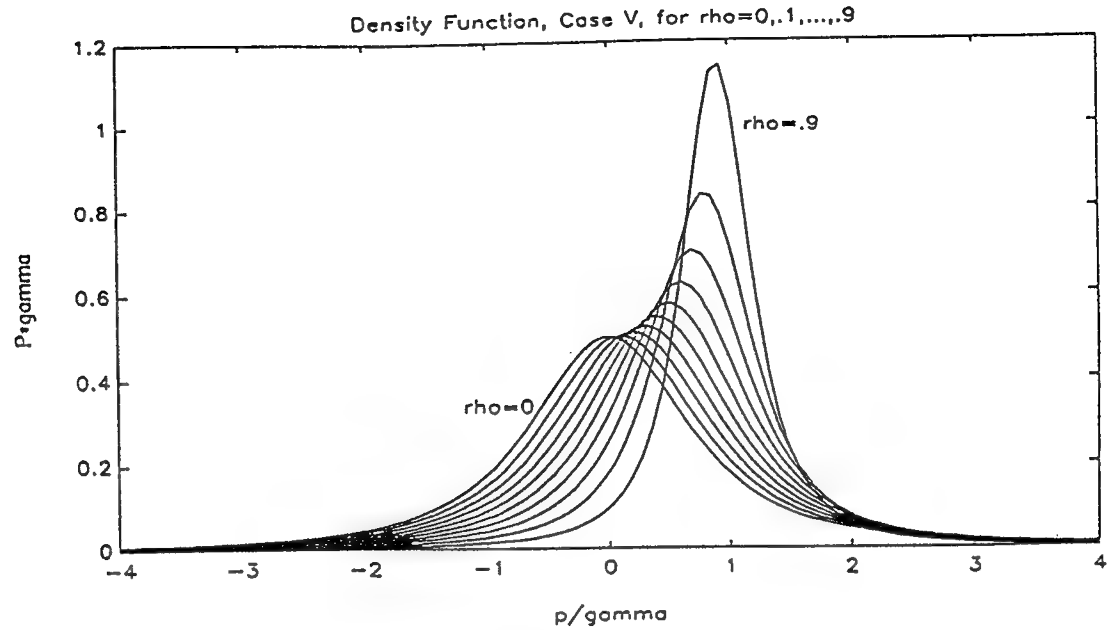

FIGURE 5-3. REAL MR DENSITY FOR ZERO MEANS, FOR VARIOUS VALUES OF THE CORRELATION COEFFICIENT

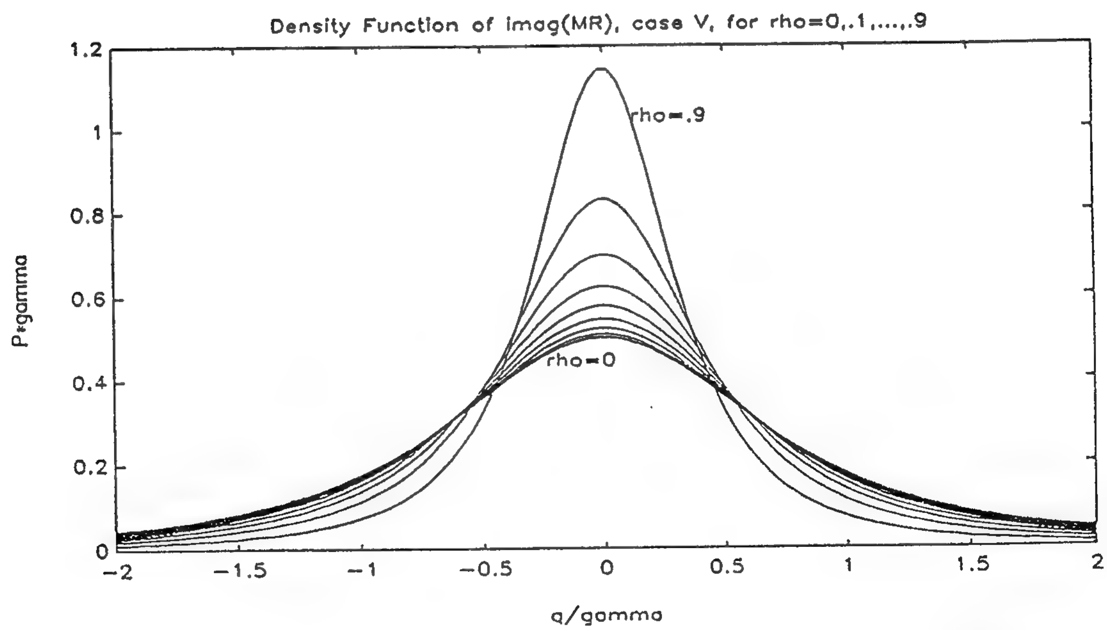

FIGURE 5-4. IMAGINARY MR DENSITY FOR ZERO MEANS, FOR VARIOUS VALUES OF THE CORRELATION COEFFICIENT

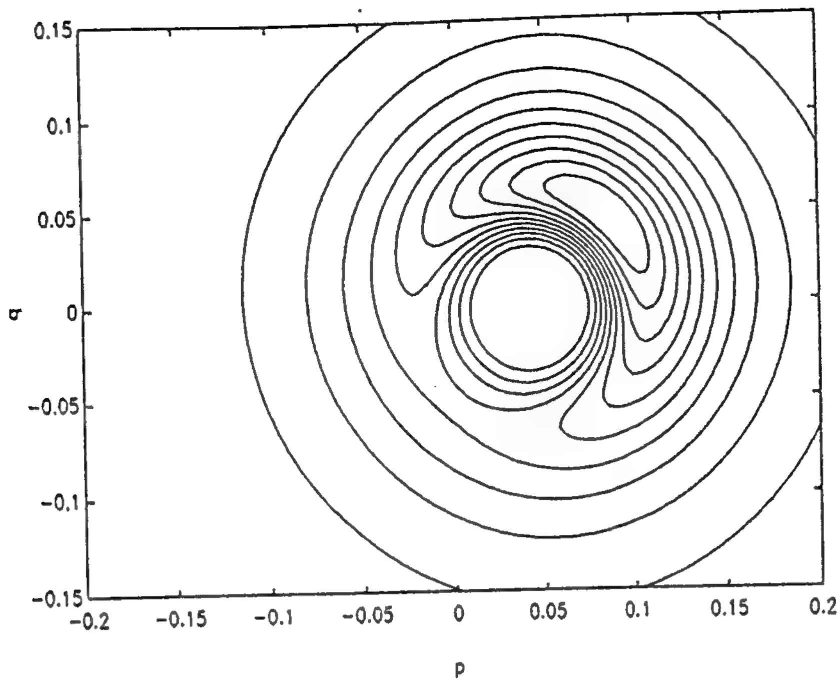

FIGURE 5-5. JOINT MR DENSITY, PARTIALLY CORRELATED NOISE, GENERAL MEANS

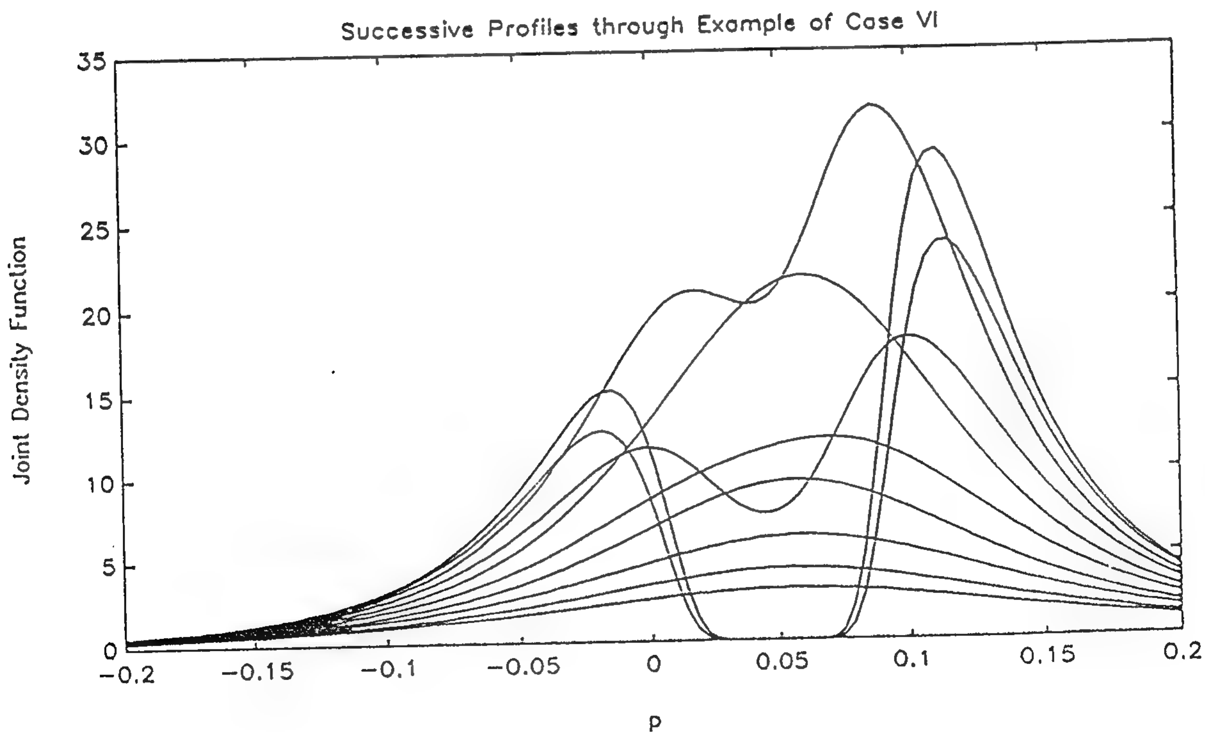

FIGURE 5-6. JOINT MR DENSITY, PARTIALLY CORRELATED NOISE, GENERAL MEANS

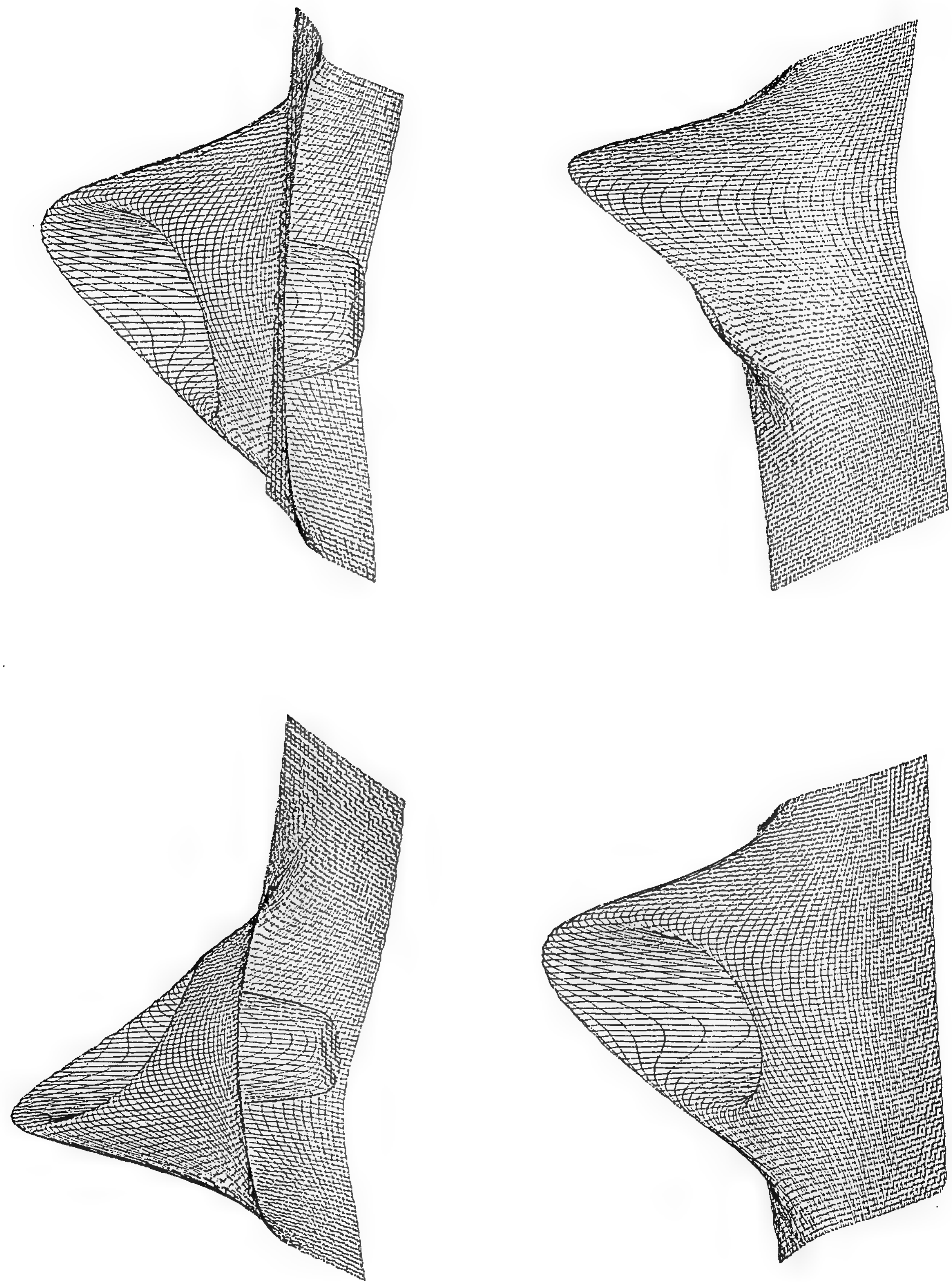

FIGURE 5-7. JOINT MR DENSITY, PARTIALLY CORRELATED NOISE, GENERAL MEANS

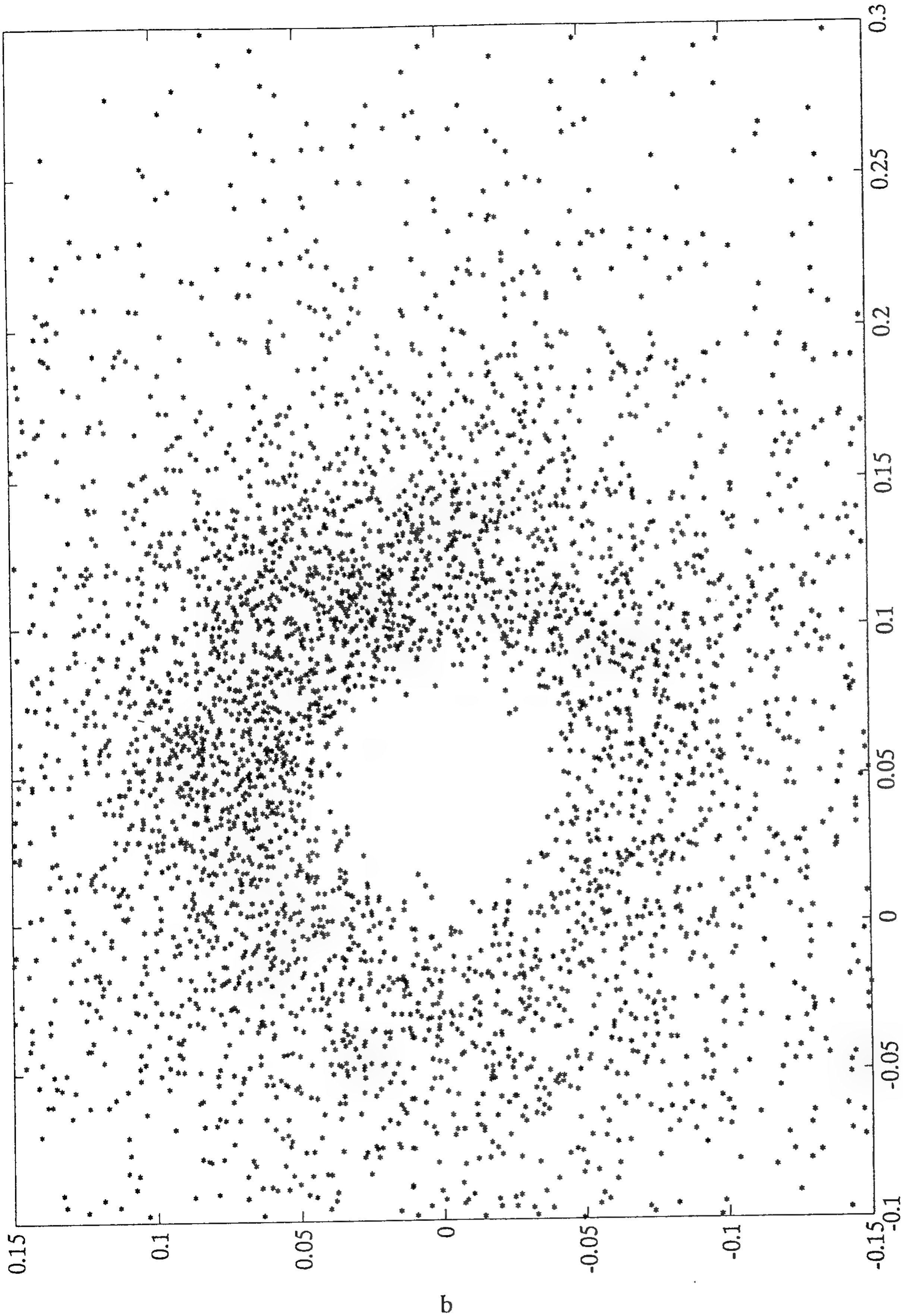

FIGURE 5-8. MONTE CARLO SCATTER DIAGRAM OF MR FOR PARTIALLY CORRELATED NOISE

CHAPTER 6

GENERAL CORRELATED NOISE

CASE VII – General Means

Here the basic assumption of gaussian density for the primary variables x, y, u, v is retained, but the covariance and mean values are arbitrary. The general four-variate gaussian density function is

$$P_{x,y,u,v}(x, y, u, v) = \frac{1}{4\pi^2 |\Sigma|^{1/2}} \exp\left[-\tfrac{1}{2}(\xi - \bar{\xi})'\Sigma^{-1}(\xi - \bar{\xi})\right] \tag{6.1}$$

where ξ stands for the column vector containing the primary variables x, y, u, v. Let the inverse of the covariance matrix be expressed by

$$\Sigma^{-1} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{12} & a_{22} & a_{23} & a_{24} \\ a_{13} & a_{23} & a_{33} & a_{34} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{pmatrix} \tag{6.2}$$

(The covariance matrix and its inverse are taken symmetrical without loss of generality.)

The transformation (2.2) is linear in r and s, and can be written as

$$\xi = \alpha r + \beta s \tag{6.3}$$

where α and β represent vectors

$$\alpha = \begin{pmatrix} p \\ q \\ 1 \\ 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} -q \\ p \\ 0 \\ 1 \end{pmatrix} \tag{6.4}$$

which are independent of r and s. With the help of these vectors the exponent in (6.1) can be written

$$\begin{aligned} &-\tfrac{1}{2}(\alpha^T\Sigma^{-1}\alpha)r^2 - (\alpha^T\Sigma^{-1}\beta)rs - \tfrac{1}{2}(\beta^T\Sigma^{-1}\beta)s^2 \\ &+ (\alpha^T\Sigma^{-1}\bar{\xi})r + (\beta^T\Sigma^{-1}\bar{\xi})s - \tfrac{1}{2}\bar{\xi}^T\Sigma^{-1}\bar{\xi} \\ &= -(Ar^2 + Brs + Cs^2 + Dr + Es + F) \end{aligned} \tag{6.5}$$

thereby defining the quantities

$$
\begin{aligned}
A &= \tfrac{1}{2}a_{11}p^2 + a_{12}pq + a_{13}p + \tfrac{1}{2}a_{22}q^2 + a_{23}q + \tfrac{1}{2}a_{33} \\
B &= -a_{11}pq + a_{12}(p^2 - q^2) - a_{13}q + a_{14}p + a_{22}pq + a_{23}p + a_{24}q + a_{34} \\
C &= \tfrac{1}{2}a_{11}q^2 - a_{12}pq - a_{14}q + \tfrac{1}{2}a_{22}p^2 + a_{24}p + \tfrac{1}{2}a_{44} \\
D &= -a_{11}\bar{x}p - a_{12}(\bar{y}p + \bar{x}q) - a_{13}(\bar{u}p + \bar{x}) - a_{14}\bar{v}p - a_{22}\bar{y}q - a_{23}(\bar{u}q + \bar{y}) \\
&\qquad - a_{24}\bar{v}q - a_{33}\bar{u} - a_{34}\bar{v} \qquad\qquad (6.6) \\
E &= a_{11}\bar{x}q - a_{12}(\bar{x}p - \bar{y}q) + a_{13}\bar{u}q - a_{14}(\bar{x} - \bar{v}q) - a_{22}\bar{y}p - a_{23}\bar{u}p \\
&\qquad - a_{24}(\bar{v}p + \bar{y}) - a_{34}\bar{u} - a_{44}\bar{v} \\
F &= \tfrac{1}{2}a_{11}\bar{x}^2 + a_{12}\bar{x}\bar{y} + a_{13}\bar{x}\bar{u} + a_{14}\bar{x}\bar{v} + \tfrac{1}{2}a_{22}\bar{y}^2 + a_{23}\bar{y}\bar{u} + a_{24}\bar{y}\bar{v} \\
&\qquad + \tfrac{1}{2}a_{33}\bar{u}^2 + a_{34}\bar{u}\bar{v} + \tfrac{1}{2}a_{44}\bar{v}^2
\end{aligned}
$$

which are independent of r and s. The joint marginal density function of p and q will now be developed. Since it is not convenient to integrate (6.1) by means of the substitution (4.4), the integration will be carried out directly in the variables r and s.

The marginal density function of the variables p, q, r is

$$P_{p,q,r}(p,q,r) = \frac{1}{4\pi^2\,|\Sigma|^{1/2}} \int_{-\infty}^{\infty} (r^2 + s^2)\exp[-Cs^2 - (E + Br)s - (Ar^2 + Dr + F)]ds \quad (6.7)$$

which becomes

$$P_{p,q,r}(p,q,r) = \frac{(\pi C)^{-3/2}e^{-G}}{8\sqrt{|\Sigma|}}[1 + 2C(r^2 + \hat{r}^2)] \qquad (6.8)$$

where the substitutions

$$\hat{r} = \frac{E + Br}{2C}, \qquad G = Ar^2 + Dr + F - C\hat{r}^2 \qquad (6.9)$$

have been made for convenience.

Equation (6.8) is meaningful only if $C > 0$, the validity of which is evident from (6.5), according to which

$$C = \tfrac{1}{2}\beta^T\Sigma^{-1}\beta \qquad (6.10)$$

As Σ (and consequently also Σ^{-1}) is positive definite in any significant situation, (6.10) must be positive for any nonvanishing β, and in particular, for the β given in (6.4).

The shape of the three-dimensional density function (6.8) can be counterintuitive. Consider the case where

$$\Sigma = \begin{pmatrix} 3.0 & -0.5 & 1.0 & 0.5 \\ -0.5 & 5.0 & -0.5 & 1.0 \\ 1.0 & -0.5 & 4.0 & 0.0 \\ 0.5 & 1.0 & 0.0 & 5.0 \end{pmatrix}, \qquad \bar{\xi} = \begin{pmatrix} 2 \\ 9 \\ -3 \\ 1 \end{pmatrix} \qquad (6.11)$$

It is difficult to display this three-dimensional function by means other than presenting two-dimensional sections accross it. Figures 6-1 through 6-4 illustrate a few of such sections.

The integration of (6.8) over r, to obtain the marginal density function of p and q, gives

$$P_{p,q}(p,q) = \frac{K e^{-M}}{2\pi\Delta^2\sqrt{-|\Sigma|\Delta}} \tag{6.12}$$

where

$$\begin{aligned} \Delta &= B^2 - 4AC \\ M &= F + \Delta^{-1}(AE^2 - BDE + CD^2) \\ K &= B^2D^2 + B^2E^2 - 4BCDE + 4C^2D^2 - 4ABDE + 4A^2E^2 + 8A^2C \\ &\quad - 2B^2C + 8AC^2 - 2AB^2 \end{aligned} \tag{6.13}$$

The steps are outlined in the Appendix. It is of interest to consider the form of the dependence of $P_{p,q}$ on p and q. The quantities Δ and K are polynomials of order four and six in (p,q), while M is a rational expression of order four over four.

Equation (6.12) is meaningful only if $\Delta < 0$. The following discussion does not pretend to be a rigorous proof that this is always the case, but tries to make it seem reasonable. In the same way that C was expressed as the quadratic form (6.10), A and B can be similarly expressed by

$$A = \tfrac{1}{2}\alpha^T\Sigma^{-1}\alpha, \qquad B = \alpha^T\Sigma^{-1}\beta \tag{6.14}$$

The quantity Δ is now expressible in terms of the vectors α and β, giving

$$\Delta = B^2 - 4AC = (\alpha^T\Sigma^{-1}\beta)^2 - (\alpha^T\Sigma^{-1}\alpha)(\beta^T\Sigma^{-1}\beta) \tag{6.15}$$

The matrix Σ^{-1} is expressible as the square of a symmetric real matrix in many, perhaps all meaningful cases. Thus, we set

$$\Sigma^{-1} = \Omega^2 \tag{6.16}$$

The requirements for the existence of the matrix Ω are not developed here, but assuming that such an Ω is available the vectors

$$a = \Omega\alpha, \qquad g = \Omega\beta \tag{6.17}$$

are defined. Then (6.15) can be written as

$$\Delta = a^T g a^T g - a^T a g^T g \tag{6.18}$$

This is equivalent to a relationship between inner vector products

$$\Delta = (\mathbf{a}\cdot\mathbf{g})^2 - |\mathbf{a}|^2|\mathbf{g}|^2 \tag{6.19}$$

which according to the Schwarz inequality cannot be positive.

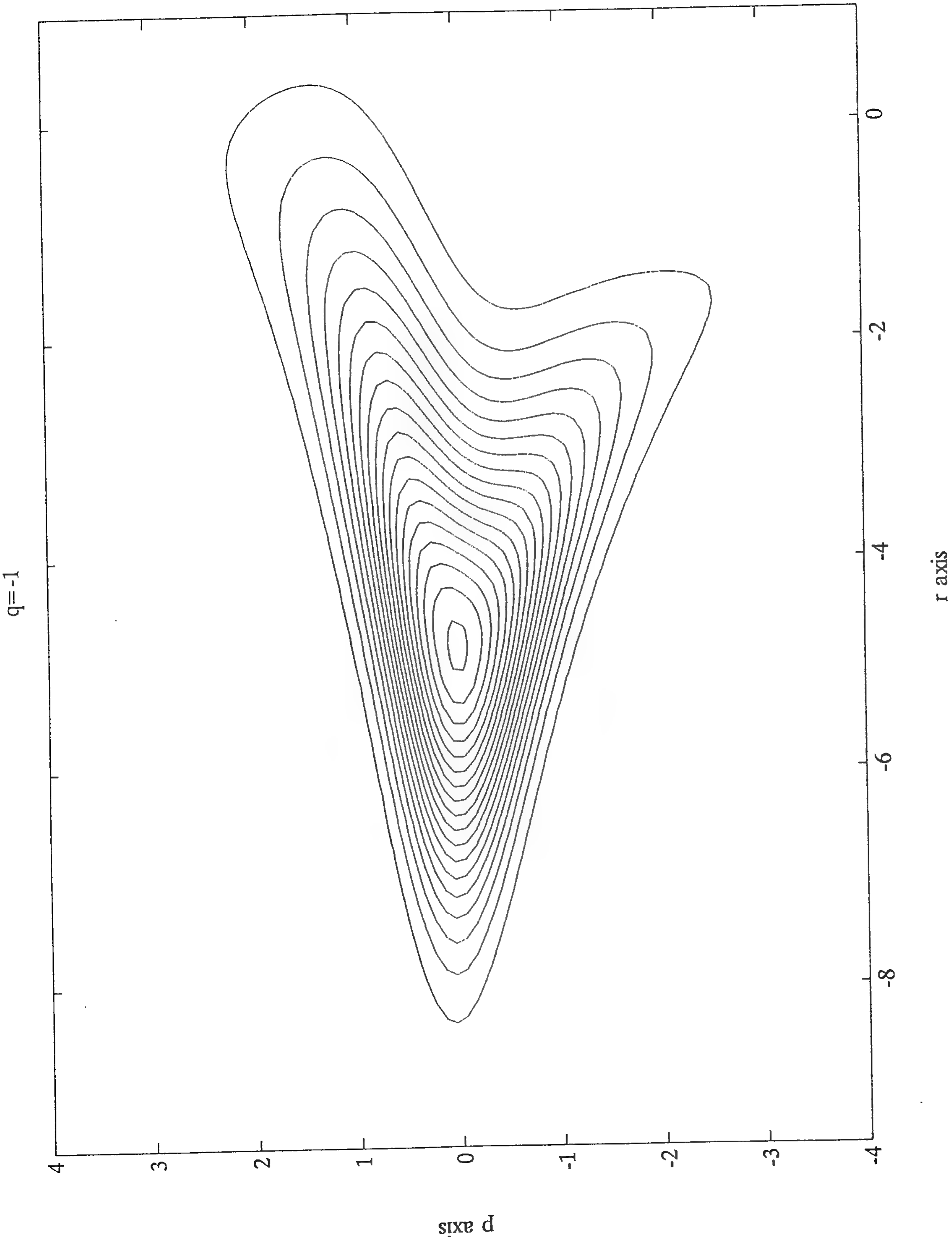

FIGURE 6-1. JOINT DENSITY OF (p, q, r) ON A q-SECTION, GENERAL CASE

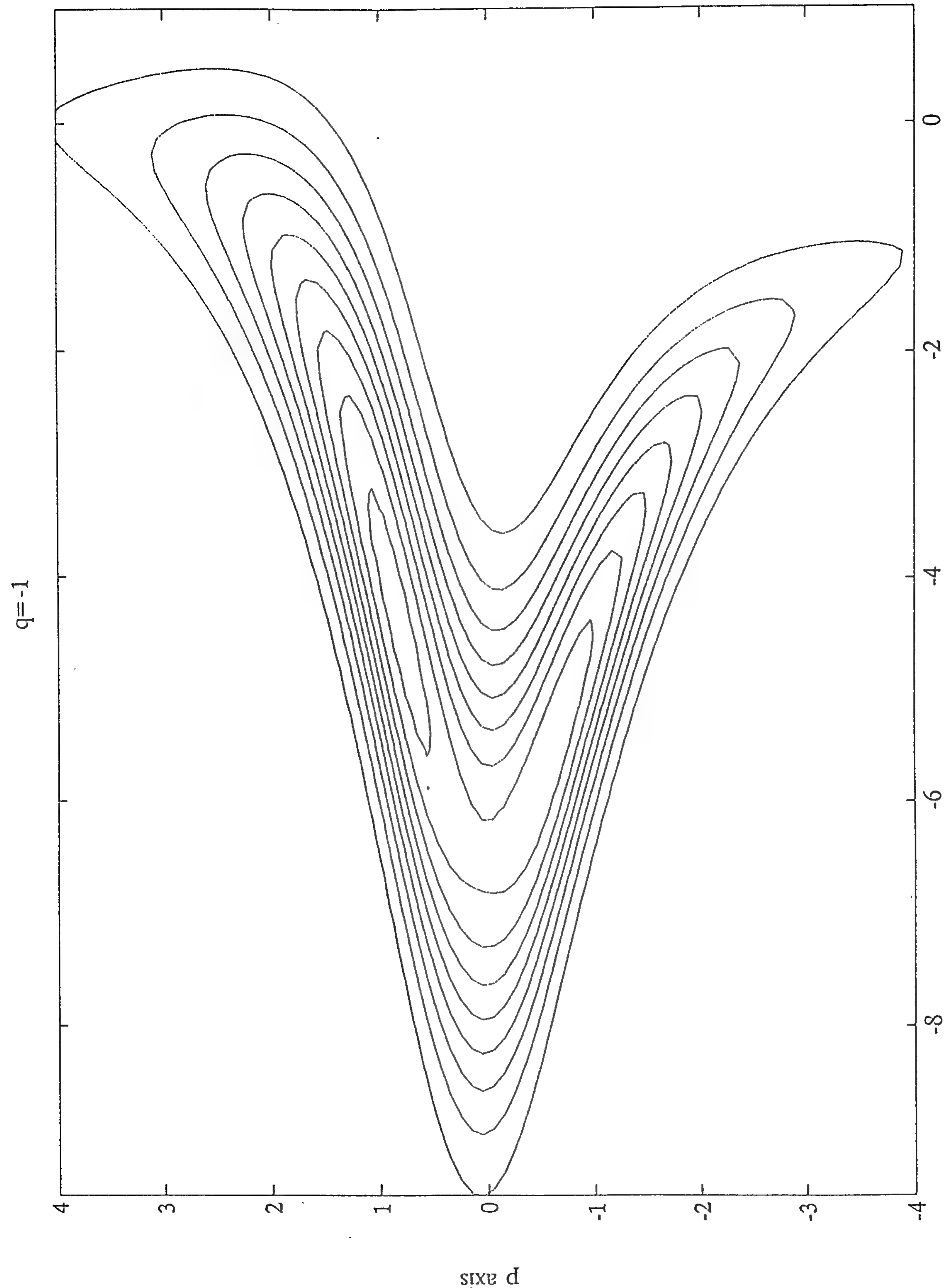

FIGURE 6-2. JOINT DENSITY OF (p, q, r) ON A q-SECTION, GENERAL CASE

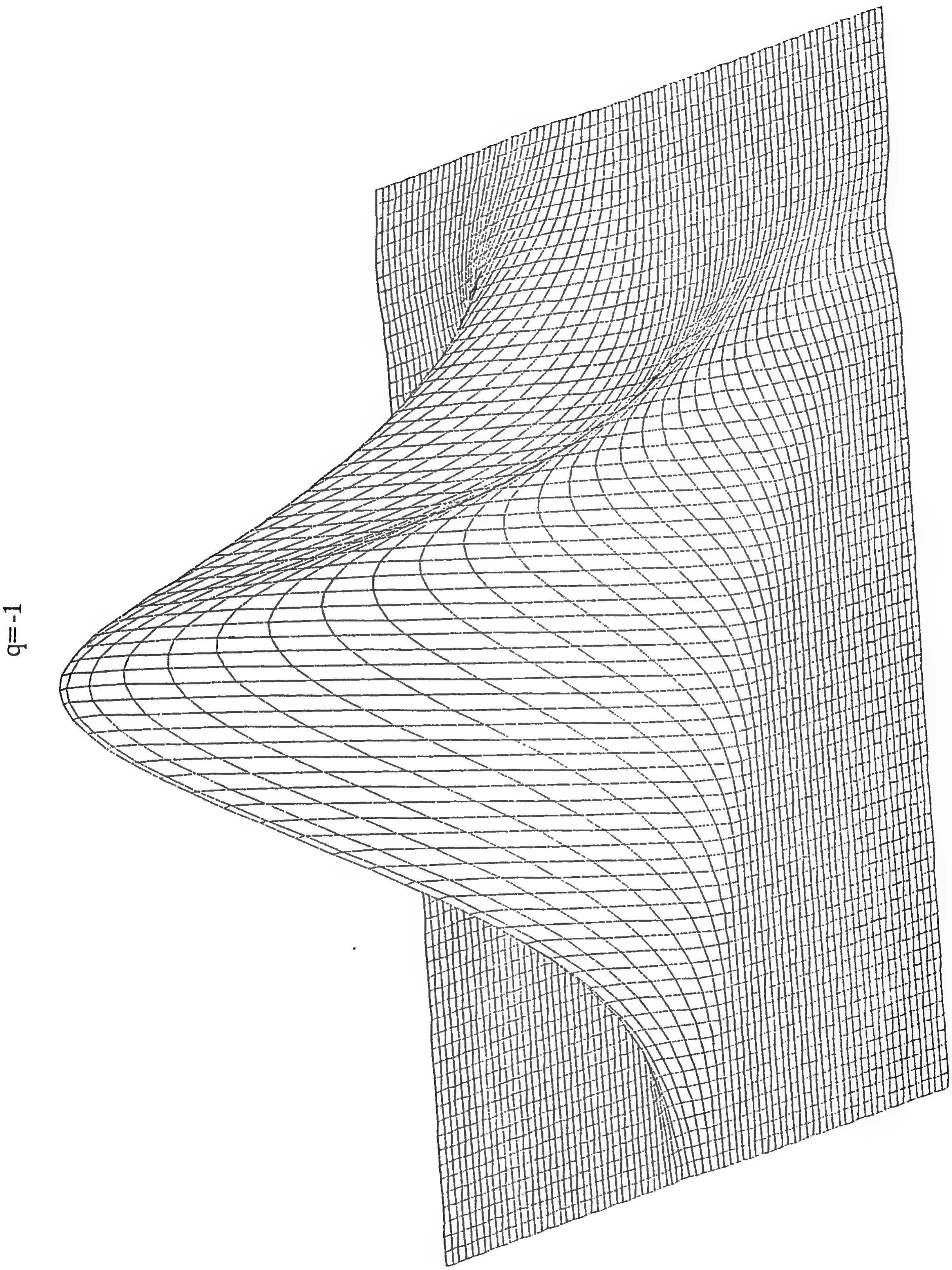

FIGURE 6-3. AN ASPECT OF THE JOINT DENSITY OF (p, q, r) ON A q-SECTION, GENERAL CASE

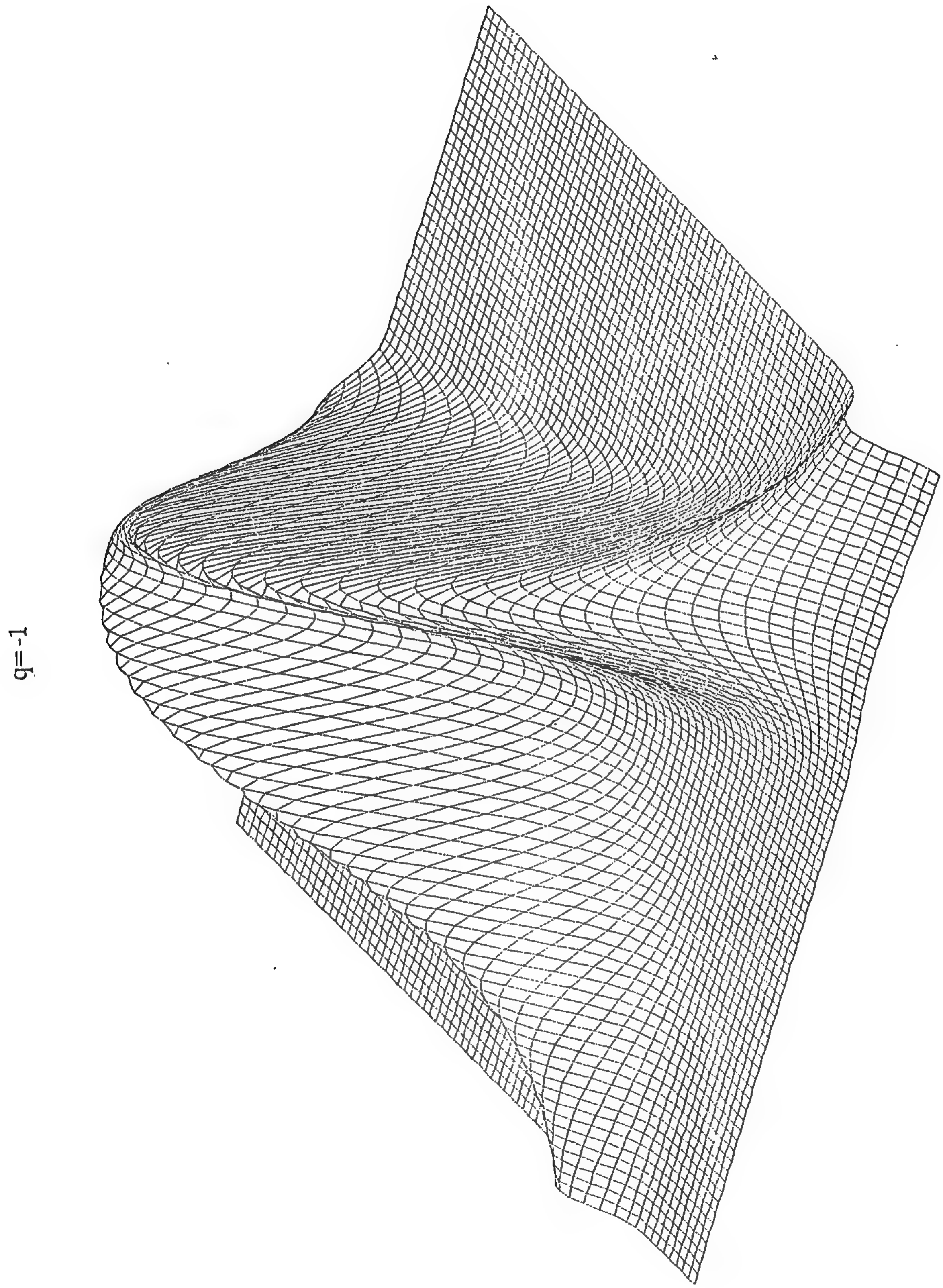

FIGURE 6-4. AN ASPECT OF THE JOINT DENSITY OF (p,q,r) ON A q-SECTION, GENERAL CASE

CHAPTER 7

THRESHHOLDING

Let us now consider threshholding the sum-channel phasors to include only those values (r,s) which obey the condition $r^2+s^2>R_0^2$. Applying this condition is cumbersome unless the density function has a rather simple form. In particular, it is straightforward to apply if the means are zero. Consider first the density function of Case I. The threshholding condition is applied by changing the limits on the outer integral of (3-5) from (0 to ∞) to (R_0 to ∞). The result is

$$\frac{1}{2\pi\sigma_x^2\sigma_u^2}\int_{R_0}^{\infty} R^3 dR \exp\left[-\left(\frac{p^2+q^2}{2\sigma_x^2}+\frac{1}{2\sigma_u^2}\right)R^2\right] \tag{7.1}$$

Performing this integration gives

$$\frac{\gamma^2}{\pi}\frac{h^2(p^2+q^2+\gamma^2)+1}{(p^2+q^2+\gamma^2)^2}\exp\left[-h^2(p^2+q^2+\gamma^2)\right] \tag{7.2}$$

where h is written for the quantity $R_0/(\sigma_x\sqrt{2})$. This expression needs to be normalized, because as it stands it will contain less than unit volume. The normalization can be applied to the modified density function of the primary variables, or directly to (7.2). The latter is more convenient, and accordingly it is found that the integral of (7.2) over all p and q is $\exp(-h^2\gamma^2)$. Dividing by this gives the final result,

$$P_{p,q}(p,q)=\frac{\gamma^2}{\pi}\frac{h^2(p^2+q^2+\gamma^2)+1}{(p^2+q^2+\gamma^2)^2}\exp\left[-h^2(p^2+q^2)\right] \tag{7.3}$$

Here also, the parameter γ can be hidden by appropriate scaling of (4.7) along with the definition $h'=\gamma h$ to give

$$p'_{p,q}(p',q')=\frac{h'^2(p'^2+q'^2+1)+1}{\pi(p'^2+q'^2+1)^2}\exp[-h'^2(p'^2+q'^2)] \tag{7.4}$$

This radially-symmetric joint density function is illustrated in Figures 7-1 and 7-2 for various threshhold values. Note that $h'=0$ corresponds to the case without threshholding for which the variance is infinite. As the threshhold value h' increases, the tails become shorter and the variance decreases. It is rather complicated to obtain the marginal distribution of p or q. However, it is seen that the means $\bar{p}$ and $\bar{q}$ are zero, while the variance is readily obtained from

$$\sigma_p^2=E[(p-\bar{p})^2]=E[p^2]=\int_{-\infty}^{\infty}dq\int_{-\infty}^{\infty}p^2P_{p,q}(p,q)dp \tag{7.5}$$

Substituting (7.3) into (7.5) gives the variance

$$\sigma_p^2 = \frac{\gamma^2}{2} e^{h^2\gamma^2} E_1(h^2\gamma^2) \tag{7.6}$$

where E_1 is the exponential integral defined by

$$E_1(z) = \int_z^\infty \frac{e^{-t} dt}{t} \qquad (|\arg z| < \pi) \tag{7.7}$$

(see Abramowitz and Stegun [3]). This is also the variance of q, as can be seen from the symmetry of (7.3). Figure 7-3 shows the variance as function of the threshhold value.

The same procedure is applied to the integral of (5-5) in the case of partially-correlated isotropic noise with zero means. After the integration of (5.5) from 0 to 2π with respect to $d\theta$, the unnormalized result is

$$\frac{1}{2\pi\sigma_x^2\sigma_u^2(1-\rho^2)} \int_{R_0}^\infty R^3\, dR \exp\left[-\frac{p^2+q^2-2\rho\gamma p+\gamma^2}{2\sigma_x^2(1-\rho^2)} R^2\right] \tag{7.8}$$

which is equal to

$$\frac{\gamma^2}{\pi}\left[\frac{h^2(p^2+q^2+\gamma^2-2\rho\gamma p)+1}{(p^2+q^2+\gamma^2-2\rho\gamma p)^2}\right] \exp\left[-h^2(p^2+q^2+\gamma^2-2\rho\gamma p)\right] \tag{7.9}$$

where h is now given by

$$h = \frac{R_0}{\sigma_x\sqrt{2(1-\rho)}} \tag{7.10}$$

The integral of (7.9) over all p and q is

$$\frac{\exp[-h^2(1-\rho^2)]}{1-\rho^2} \tag{7.11}$$

The normalized density function is (7.9) divided by (7.11), giving

$$P_{p,q}(p,q) = \frac{\gamma^2(1-\rho^2)}{\pi}\left[\frac{h^2(p^2+q^2+\gamma^2-2\rho\gamma p)+1}{(p^2+q^2+\gamma^2-2\rho\gamma p)^2}\right] \exp[-h^2(p^2+q^2+\rho^2\gamma^2-2\rho\gamma p)] \tag{7.12}$$

or

$$P'_{p,q}(p',q') = \frac{1-\rho^2}{\pi}\left[\frac{h'^2(p'^2+q'^2+1-2\rho p')+1}{(p'^2+q'^2+1-2\rho p')^2}\right] \exp[-h'^2(p'^2+q'^2+\rho^2-2\rho p')] \tag{7.13}$$

This joint density function is illustrated in Figure 7-4. It is seen by inspection that $\bar{q} = 0$. To obtain the mean value $\bar{p}$ the substitutions $t = p - \rho\gamma$ or $t' = t/\gamma = p' - \rho$ and $t = R\cos\theta$, $q = R\sin\theta$ are made in (7.13). Thus,

$$P'_{p,q} = \frac{1-\rho^2}{\pi}\left[\frac{h'^2(t'^2+q'^2+1-\rho^2)}{(t'^2+q'^2+1-\rho^2)^2}\right] \exp[-h'^2(t'^2+q'^2)] \tag{7.14}$$

or

$$P'_{p,q} = \frac{1-\rho^2}{\pi}\left[\frac{h'^2(R^2+1-\rho^2)+1}{(R^2+1-\rho^2)^2}\right]\exp(-h'^2R^2) \tag{7.15}$$

This function is shown in Figure 7-5. The mean value $\bar{t}'$ is zero, and hence $\bar{p} = \rho\gamma$, because (7.14) is an even function of t' and the integral defining its mean converges. It may seem somewhat surprising that $\bar{p}$ is independent of h.

Again, the marginal distributions of p and of q are rather cumbersome, and are not developed here. The variances of p and of q, on the other hand, are easily obtained. Thus,

$$E[t^2] = \frac{\gamma^2(1-\rho^2)}{\pi}\int_0^{2\pi}\cos^2\theta d\theta\int_0^{\infty}R^3dR\left[\frac{h^2[R^2+\gamma^2(1-\rho^2)]+1}{[R^2+\gamma^2(1-\rho^2)]^2}\right]\exp[-h^2R^2] \tag{7.16}$$

which is given by

$$E[t^2] = \frac{\gamma^2(1-\rho^2)}{2}\exp[-h^2\gamma^2(1-\rho^2)]E_1[h^2\gamma^2(1-\rho^2)] \tag{7.17}$$

The same result is obtained for $E[q^2]$. The variance of p is then

$$E[\langle p^2\rangle = \langle t^2\rangle + \rho^2\gamma^2, \qquad \text{or} \quad E[(p-\bar{p}^2)] = E[t^2] \tag{7.18}$$

The effect of threshholding can be evaluated by analysis only in the simplest cases. When analytic solutions are not available or are not feasible to obtain, Monte Carlo methods can be used in the manner described in Chapter 3. Rather severe threshholding, with $R_0 = 36$, was applied to the example of Case IV presented in Figure 4-10. The result is shown in Figure 7-6. The effect of threshholding is seen quite dramatically by comparing the two figures. Not surprisingly, the spread of points in the threshholding example appears much reduced, and in fact, a variance exists and can be estimated.

A similar study of threshholding with the rather high value of $R_0 = 36$ was done on the example of Case VI presented in Figure 5-8. The result, shown in Figure 7-7, displays a noteworthy ring structure. These examples may be somewhat atypical, as they were selected in order to exhibit such curious features. The marginal distribution of p corresponding to Figure 7-7, obtained by 330,000 Monte Carlo experiments, is shown in Figure 7-8. It is noted that the deep hole, quite conspicuous on the bivariate distribution, is much less prominent on the marginal distribution.

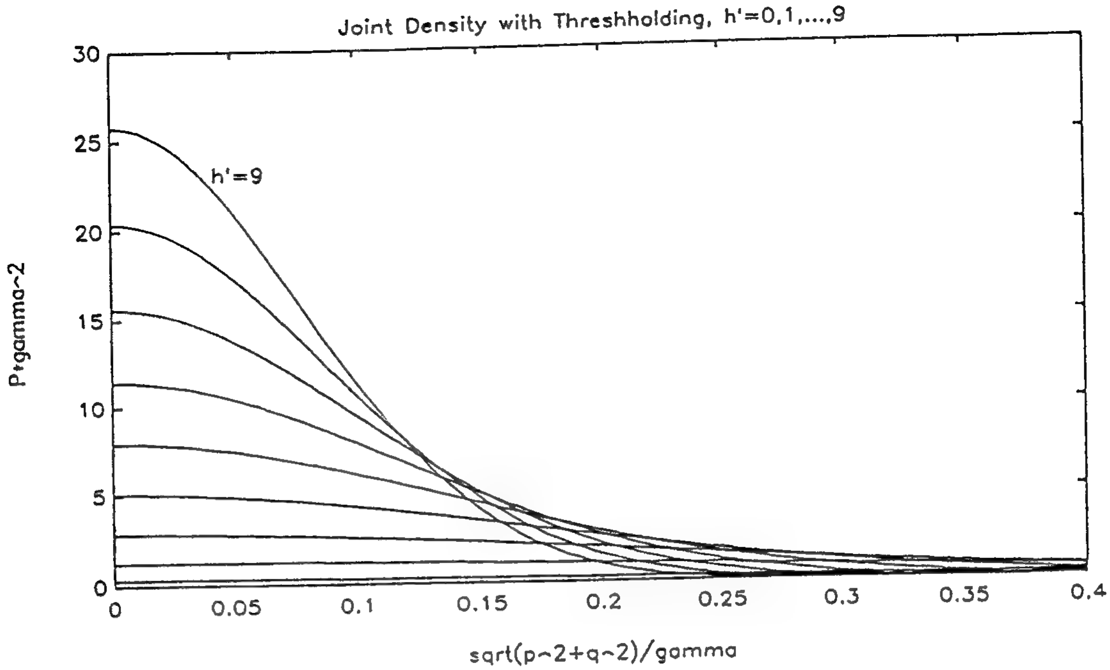

FIGURE 7-1. JOINT MR DENSITY WITH VARIOUS THRESHHOLDS

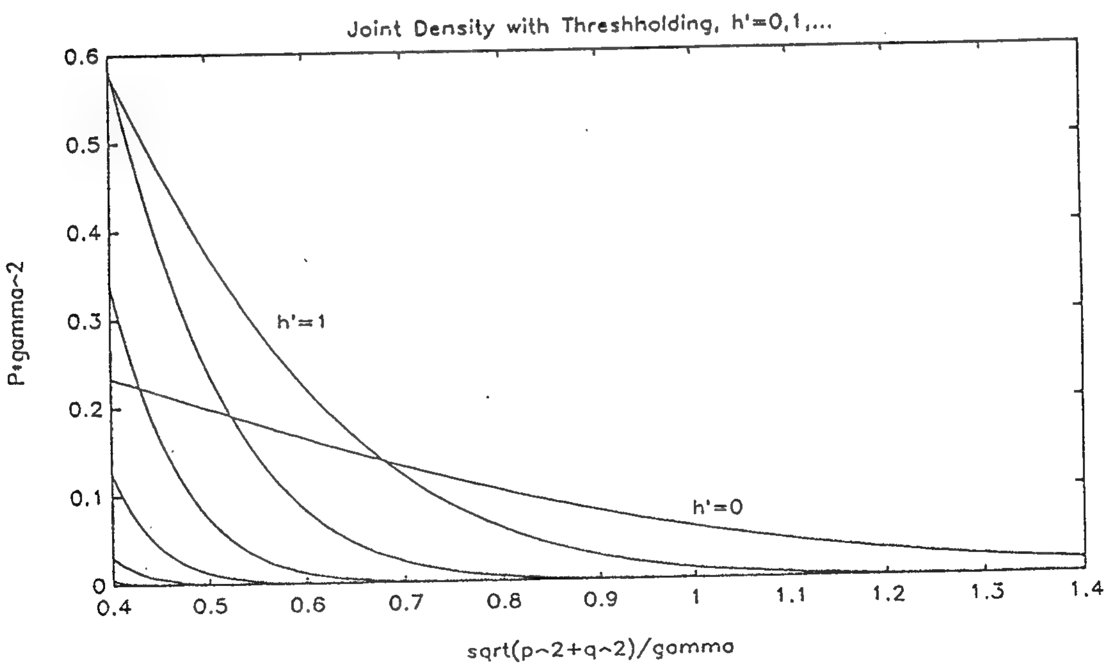

FIGURE 7-2. JOINT MR DENSITY WITH VARIOUS THRESHHOLDS

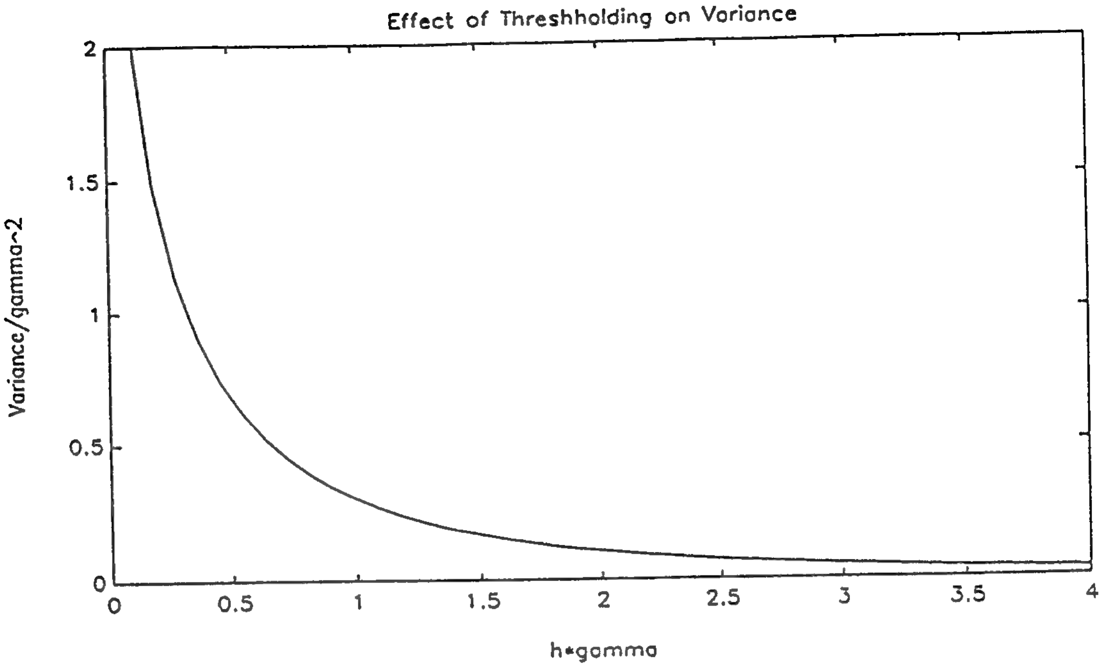

FIGURE 7-3. VARIANCE OF REAL MR AS FUNCTION OF THRESHHOLD VALUE

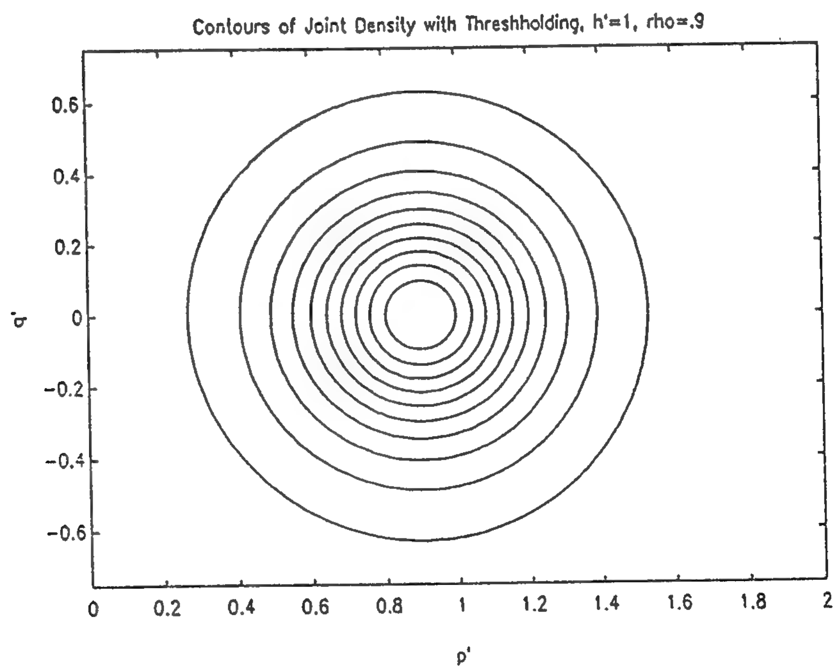

FIGURE 7-4. JOINT MR DENSITY WITH THRESHHOLDING AND PARTIAL CORRELATION OF NOISE

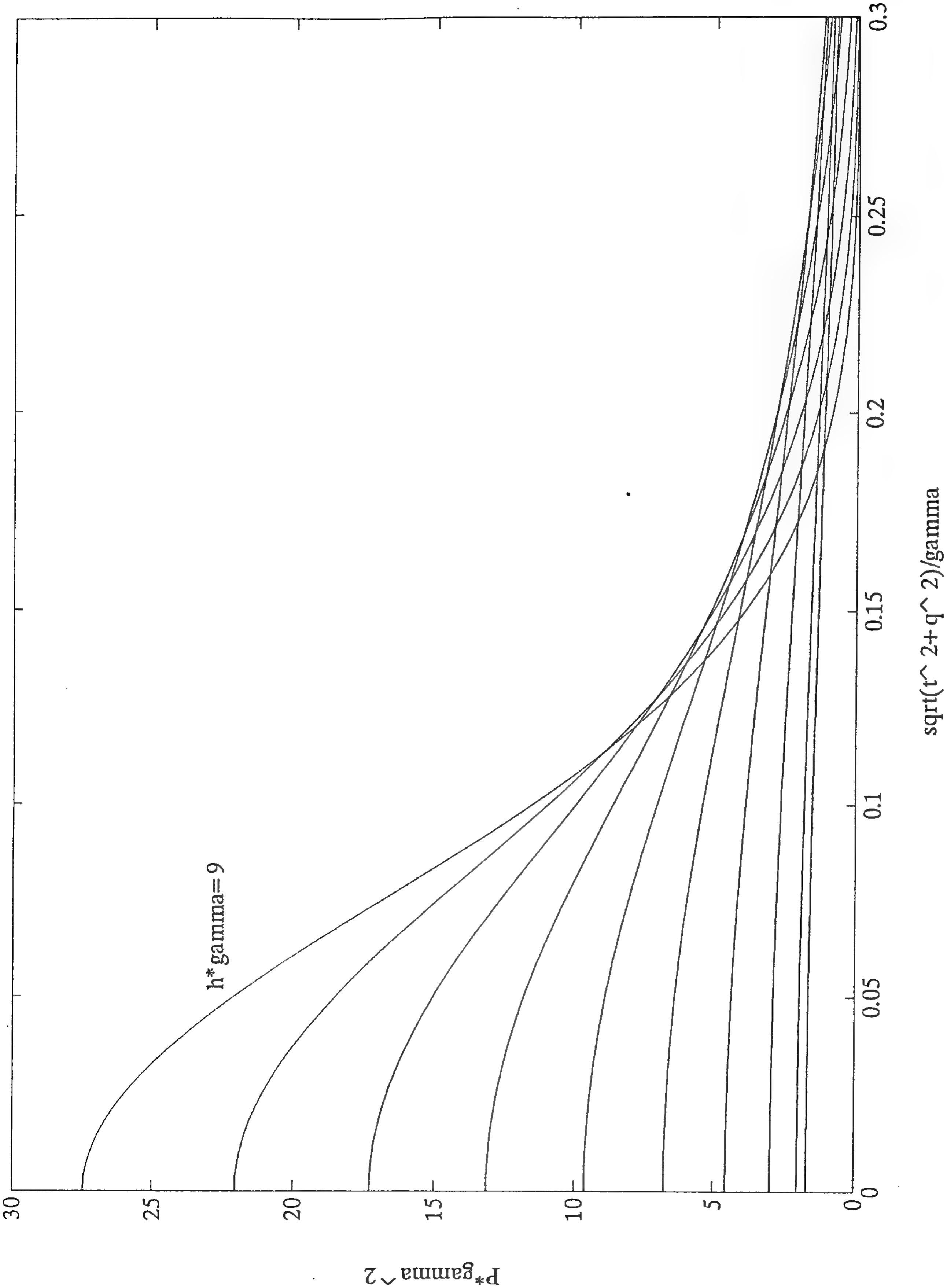

FIGURE 7-5. JOINT MR DENSITY WITH VARIOUS THRESHOLDS ON THE SUM PHASOR, PARTIAL NOISE CORRELATION

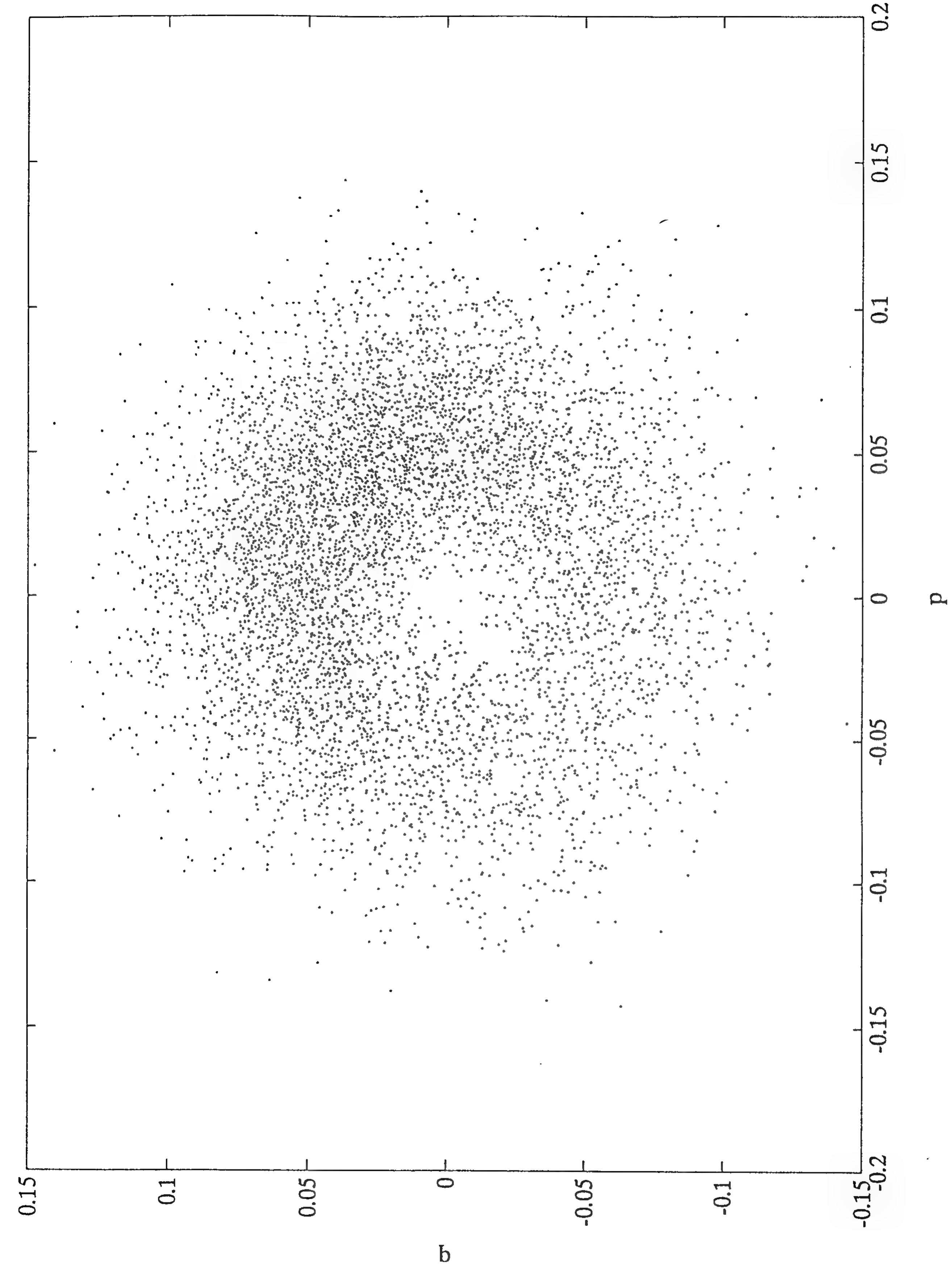

FIGURE 7-6. MONTE CARLO SCATTER DIAGRAM OF JOINT MR DENSITY, UNCORRELATED NOISE WITH THRESHHOLD

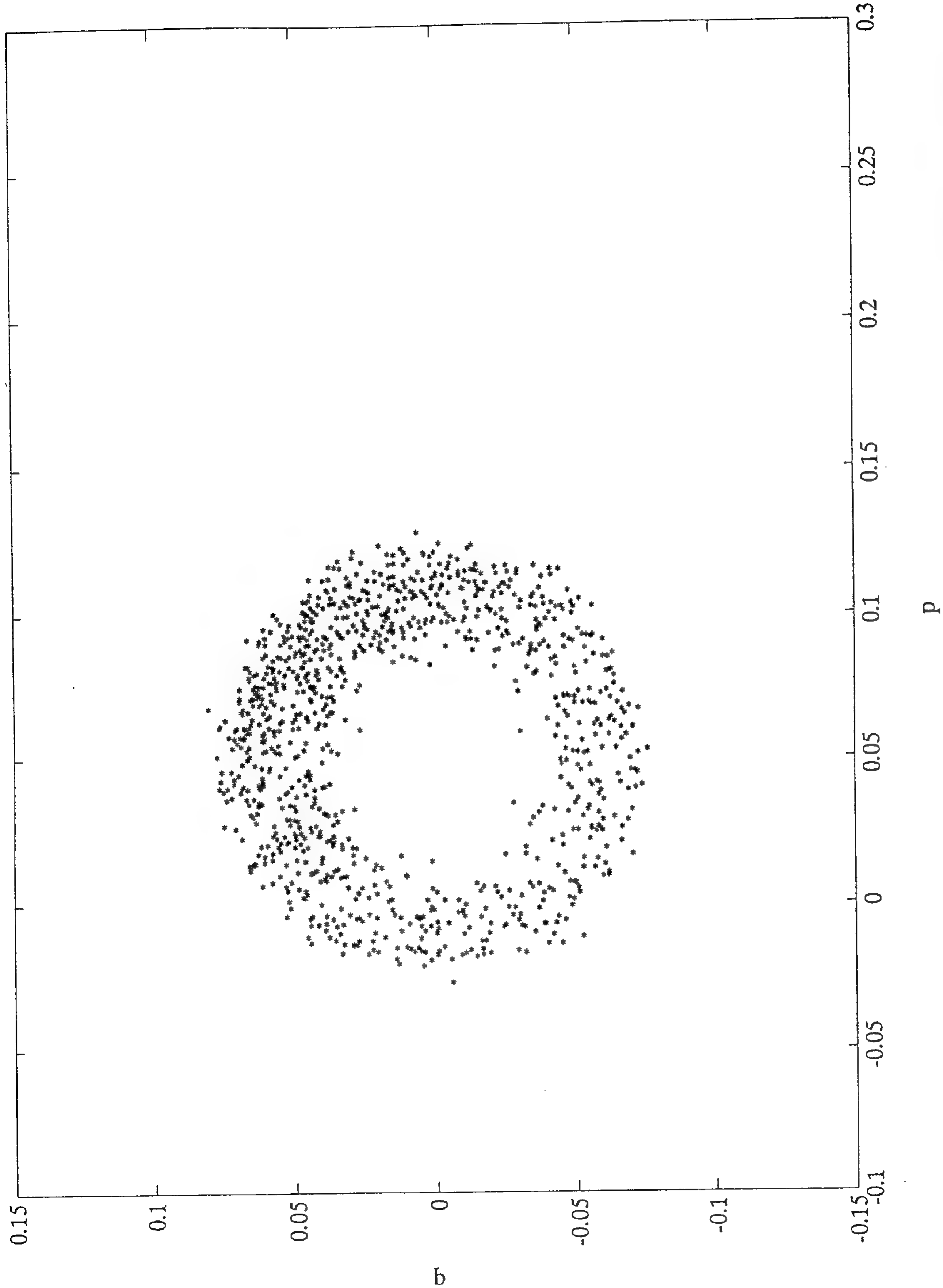

FIGURE 7-7. MONTE CARLO SCATTER DIAGRAM OF JOINT MR DENSITY FOR PARTIALLY CORRELATED NOISE AND THRESHOLD

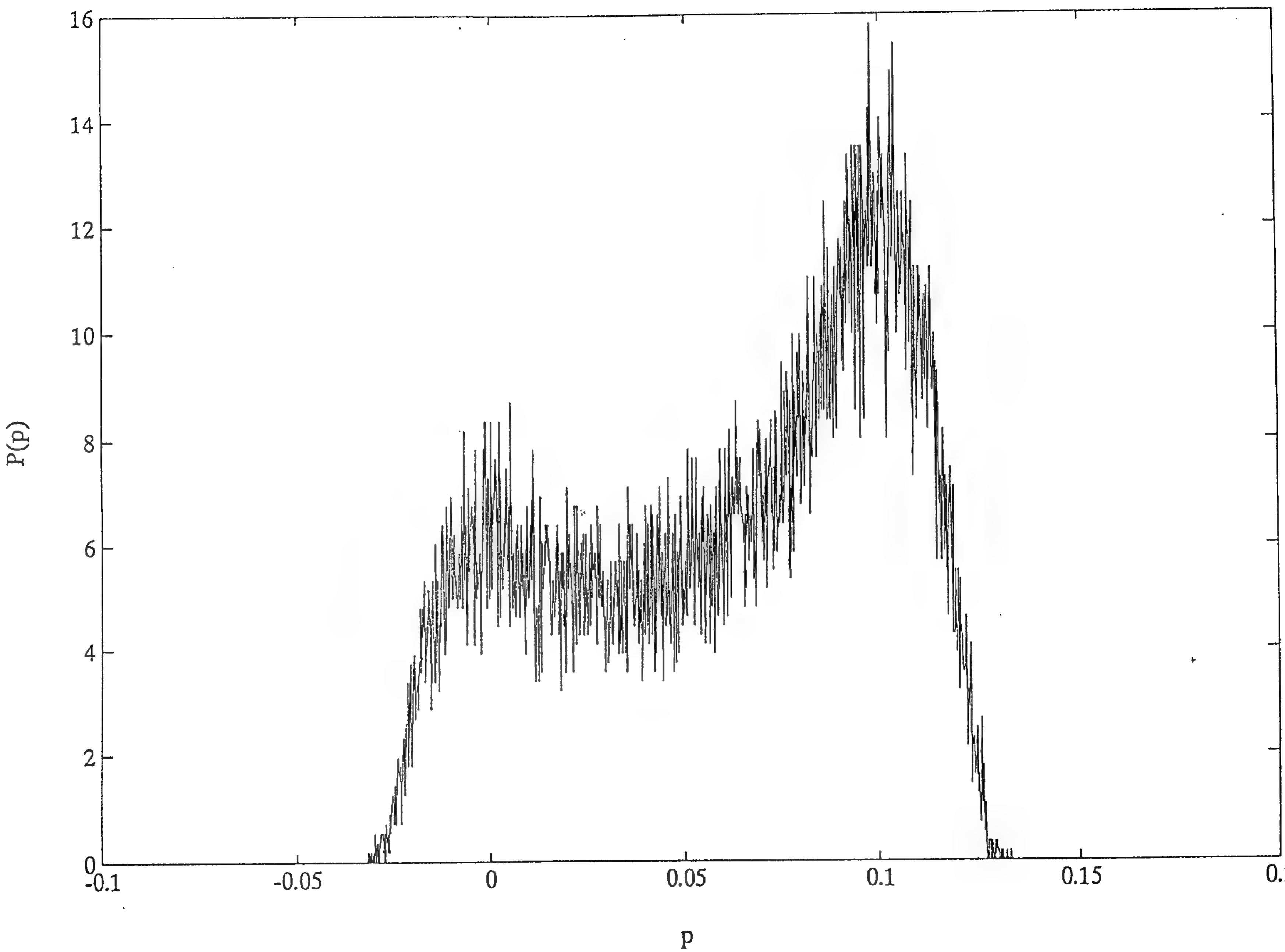

FIGURE 7-8. MARGINAL DISTRIBUTION OF REAL(MR) FOR PARTIALLY CORRELATED NOISE AND THRESHOLD, DETERMINED BY MONTE CARLO METHOD

CHAPTER 8

CONCLUSIONS

The joint density function of the real and imaginary parts of the monopulse ratio have been developed for various models of the Gaussian joint density of the sum and difference phasors of a monopulse radar. The resulting functions have been reduced to a particularly simple form by introducing alternative parameters defining the joint density of the primary variables (i.e., the parameters X, U, W, and Y of Chapter 4). It was shown that the dependence on the noise level in the sum and difference phasors is usually representable in terms of a single parameter, γ, equal to the ratio of the two standard deviations. A further simplification in form was achieved by scaling with factors dependent on γ to eliminate this parameter from the expressions defining the joint densities.

In a few cases it was possible to obtain simple expressions for the marginal density of the real (or imaginary) part of the MR. Results were presented for the cases of uncorrelated, isotropic noise with zero means of the primary phasors (Case I), and zero mean of the difference phasor (Case II), and for partially-correlated isotropic noise with zero means of the primary phasors (Case V). The term "partially-correlated" is used here to denote the case where the covariance matrix contains some zero elements, and does not refer to the degree of correlation or to the value of the correlation coefficient. Marginal densities of the real part of the MR have been obtained in other contexts by [1,2,3,4].

Generally, the marginal density functions of a sole component of the MR are more complicated and in many cases less interesting than the complete joint density. An expression for the mean value of the MR was obtained in most of the cases. The MR variance, known to be infinite usually for an observation consisting of a single pulse, was obtained only for the cases resulting in a finite value, namely, where a threshhold on the sum-phasor magnitude was imposed (in Chapter 7). Any positive threshhold value results in a finite MR variance.

Some of the most interesting solutions are those resulting from non-vanishing means of the primary variables. Although scintillation and phase uncertainty tend to produce zero-mean density functions of the primary variables, the consequence of non-zero means may be applicable to some situations. One such case is that of observations on a target located at a specific point. Of course, the analysis is much simpler if the mean value of either the sum or the difference phasor is zero.

Three basic models of noise covariance were studied: uncorrelated isotropic noise (Chapter

4), partially-correlated isotropic noise (Chapter 5), and a general noise covariance matrix (Chapter 6). The effect of the correlation coefficient on the joint MR density was studied. Monte Carlo simulations were used to corroborate the analytically-obtained density functions in a couple of cases. A feature that has not been previously noted in the literature is a prominent minimum or 'hole' in the joint MR density that occurs under some circumstances. Such a minimum has been observed in cases of non-zero means of the primary phasors (as in Figures 4-7 through 4-10 and 5-5 through 5-8). Some new definite integrals that may be useful in other studies were evaluated and are given in (A.33), (A.35), and (A.36).

In carrying out this study it became apparent to the authors that a better understanding of noise processes in monopulse radar systems is needed. The various noise sources affect the statistical properties of the density functions in different ways. To our knowledge, the existing literature lacks a simple and direct description of the noise processes. Development of a noise model which is based on the particular features of each noise source is planned to establish a small number of parameters which characterize the complete noise environment with sufficient accuracy for practical work.

REFERENCES

1. Kanter, I., 1977: "The probability density function of the monopulse ratio for N looks at a combination of constant and Rayleigh targets," IEEE Trans. on Information Theory, IT-23, no. 5, 643-648.

2. Tullsson, B., 1991: "Monopulse Tracking of Rayleigh Targets: A Simple Approach," IEEE Transactions on Aerospace and Electronic Systems, Vol. 27 No. 3, pp 520-531.

3. Seifer, A. D., 1992: "Monopulse-Radar Angle Tracking in Noise or Noise Jamming." IEEE Trans. on Aerospace and Electronic Systems, Vol. 28, No. 3, pp 622-637.

4. Seifer, A. D., 1994: "Monopulse-Radar Angle Measurement in Noise," IEEE Trans. on Aerospace and Electronic Systems, Vol. 30, No. 3, 950-957.

5. Mood, A. M., F. A. Graybill, and D. C. Boes: "Introduction to the Theory of Statistics", third ed., McGraw-Hill Series in Probability and Statistics, 564 pp., New York, 1974.

6. Abramowitz, M. and I. A. Stegun, 1971: "Handbook of Mathematical Functions," Dover, NY.

7. Gradshsteyn, I. S. and Ryzhik, I. M., 1980: "Table of Integrals, Series, and Products," Academic Press, NY, 1160 pp.

APPENDIX A

EVALUATION OF INTEGRALS

The integration of (4.5) is carried out by the substitution $w = R^2$, which gives

$$P_{p,q}(p,q) = \frac{1}{4\pi\sigma_x^2\sigma_u^2}\int_0^\infty \exp\left[-\left(\frac{p^2+q^2}{2\sigma_x^2}+\frac{1}{2\sigma_u^2}\right)w\right] w\,dw \tag{A.1}$$

which gives (4.6).

The integral (4.9) is evaluated by Abramowitz and Stegun ([3], equation 3.3.24) who give the indefinite integral

$$\int \frac{dx}{(x^2+a^2)} = \frac{1}{2a^3}\arctan\left(\frac{x}{a}\right) + \frac{x}{2a^2(x^2+a^2)} \tag{A.2}$$

By setting $a^2 = p^2 + \gamma^2$, the last term vanishes at both limits yielding the result indicated in (4.9).

The integration of (4.15) is carried out by a translation along the θ axis by expressing the integrand in the form

$$\exp[C\cos(\theta+\alpha)] = \exp[C(\cos\alpha\cos\theta - \sin\alpha\sin\theta)] \tag{A.3}$$

where C and α are to be determined. Evidently they must satisfy the relations

$$\begin{aligned} C\cos\alpha &= R\bar{u}/\sigma_u^2 \\ C\sin\alpha &= -R\bar{v}/\sigma_u^2 \end{aligned} \tag{A.4}$$

which determine the value of C to be

$$C = \frac{R\sqrt{\bar{u}^2+\bar{v}^2}}{\sigma_u^2} \tag{A.5}$$

A transformation is now made to a new variable of integration,

$$\phi = \theta + \alpha \tag{A.6}$$

The angle α can similarly be determined, but its value is irrelevant as the integration is over a complete cycle of either θ or ϕ. The result is

$$\int_0^{2\pi} \exp[C\cos\phi]d\phi \tag{A.7}$$

which has the value

$$2\pi I_0(C) = 2\pi I_0\left(\frac{R\sqrt{\bar{u}^2+\bar{v}^2}}{\sigma_x^2}\right) \tag{A.8}$$

as given by [3] (equation 9.6.19).

The authors are indebted to John E. Gray for giving direction in the evaluation of this integral. The integral of (4.17) is carried out by expressing the integral in the form

$$\int_0^\infty R^3 e^{-AR^2} I_0(BR)dR \tag{A.9}$$

where A and B are constants with respect to the integration, and are readily evaluated in terms of the parameters in (4.17). Equation (9.6.10) of [3] gives the following series expansion for the Modified Bessel Function,

$$I_0(z) = \sum_{k=0}^{\infty} \frac{(z^2/4)^k}{[k!]^2} \tag{A.10}$$

Substitution into (A.9) gives

$$\sum_{k=0}^{\infty} \frac{2^{-2k} B^{2k}}{[k!]^2} \int_0^{\infty} R^{2k+3} e^{-AR^2} dR \tag{A.11}$$

The integral in the above expression can be evaluated by making the substitution $z = AR^2$, which gives

$$\int_0^{\infty} R^{2k+3} e^{-AR^2} dR = \frac{1}{2A^{k+2}} \int_0^{\infty} z^{k+1} e^{-z} dz \tag{A.12}$$

The integral part of (A.12) is just $(k+1)!$, as can be found in any standard integral table. Substituting back into (A.11) gives

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{2^{-2k} B^{2k} (k+1)!}{2[k!]^2 A^{k+2}} &= \frac{1}{2A^2} \sum_{k=0}^{\infty} \frac{(k+1)}{k!} \left(\frac{B^2}{4A}\right)^k \\ &= \frac{1}{2A^2} \left[\left(\frac{B^2}{4A}\right) \sum_{k=1}^{\infty} \frac{1}{(k-1)!} \left(\frac{B^2}{4A}\right)^{k-1} + \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{B^2}{4A}\right)^k \right] \\ &= \frac{1}{2A^2} \exp\left(\frac{B^2}{4A}\right) \left[\left(\frac{B^2}{4A}\right) + 1 \right] \end{aligned} \tag{A.13}$$

Considering (4.18) to show that the integral over all p and q is unity, first make the substitution

$$\begin{aligned} p &= R\cos\theta \\ q &= R\sin\theta \end{aligned} \tag{A.14}$$

Thus,

$$\int_{-\infty}^{\infty} dq \int_{-\infty}^{\infty} P_{p,q}(p,q)dp = \int_0^{\infty} R\,dR \int_0^{2\pi} P_{p,q}(p,q)d\theta = 2\pi \int_0^{\infty} R\,dR\, P_{p,q}(p,q) \tag{A.15}$$

To facilitate the integration with respect to R, μ is used as the variable of integration, where

$$\mu = \frac{U\gamma^2}{R^2 + \gamma^2}, \qquad R\,dR = -\frac{U\gamma^2}{2\mu^2} d\mu \tag{A.16}$$

The integral then becomes

$$\frac{e^{-U}}{U} \int_0^{U} (1+\mu)\, e^{\mu} d\mu \tag{A.17}$$

which is unity.

To evaluate the integral (4.22) the exponential is expressed as a power series and the integration is performed term by term. For convenience a change of variable is made so that the quadratic term in the radical will have a coefficient of unity. Thus,

$$z = \mu\sqrt{p^2+\gamma^2} \tag{A.18}$$

Evaluation of each term in the resulting series requires the integral

$$\int_0^{2A} \frac{z^n\,dz}{\sqrt{2Az-z^2}} \tag{A.19}$$

for $n = 1, 2, \ldots$. The integral exists even though the denominator in the integrand vanishes at both limits. The result is

$$\frac{\pi(2n-1)!A^n}{2^{n-1}(n-1)!n!} \tag{A.20}$$

When the exponential factor is included, the result can be expressed in series form. Thus,

$$\begin{aligned} \int_0^{2A} \frac{z^n e^{Bz}\,dz}{\sqrt{2Az-z^2}} &= \sum_{k=0}^{\infty} \frac{B^k}{k!} \int_0^{2A} \frac{z^{n+k}\,dz}{\sqrt{2Az-z^2}} \\ &= 2\pi\left(\frac{A}{2}\right)^n \sum_{k=0}^{\infty} \frac{(2n+2k-1)!(AB)^k}{2^{n+k}k!(n+k-1)!(n+k)!} \end{aligned} \tag{A.21}$$

With this result the integral (4.22) can be readily evaluated.

To show that the density function $P_p(p)$ of (4.23) encloses unit area, the integral

$$\int_{-\infty}^{\infty} \frac{dp}{(p^2+\gamma^2)^{n+\frac{1}{2}}} \tag{A.22}$$

is needed. With the substitution

$$p = \gamma\tan\theta, \qquad dp = \gamma\sec^2\theta\,d\theta \tag{A.23}$$

the integral can be written

$$2\int_0^{\pi/2} \frac{\gamma\sec^2\theta\,d\theta}{\gamma^{2n+1}(\tan^2\theta+1)^{n+\frac{1}{2}}} = 2\gamma^{-2n}\int_0^{\pi/2} \cos^{2n-1}\theta\,d\theta \tag{A.24}$$

The definite integral occurring in (A.24) is found in any standard table of definite integrals, with the final result being

$$2\gamma^{-2n}\left[\frac{2\cdot4\cdot6\cdots(2n-2)}{1\cdot3\cdot5\cdots(2n-1)}\right] = \frac{2^{2n-1}[(n-1)!]^2\gamma^{-2n}}{(2n-1)!} \tag{A.25}$$

The expression (4.23) can be written

$$\int_{-\infty}^{\infty} P_p(p)dp = \frac{\gamma^2 e^{-U}}{2} \sum_{j=0}^{\infty} \frac{(2j+1)!}{[j!]^3} \left(\frac{U\gamma^2}{4}\right)^j \int_{-\infty}^{\infty} \frac{dp}{(p^2+\gamma^2)^{j+1+\frac{1}{2}}} \tag{A.26}$$

When (A.25) is substituted into (A.26) the integral can be written

$$\int_{-\infty}^{\infty} P_p(p)dp = e^{-U} \sum_{k=0}^{\infty} \frac{U^j}{j!} = 1 \tag{A.27}$$

An alternative formulation in terms of standard functions is obtained with the help of formula No. 3.364.1 of [4], which states that

$$\int_0^2 \frac{e^{-px}dx}{\sqrt{x(2-x)}} = \pi e^{-p} I_0(p) \quad \text{for} \quad p > 0 \tag{A.28}$$

It can be demonstrated that the restriction on p can be relaxed to include all real values. With the substitution $\mu = 2 - x$ the integral (A.28) becomes

$$e^{-2p} \int_0^2 \frac{e^{p\mu}d\mu}{\sqrt{\mu(2-\mu)}} = \pi e^{-p} I_0(p) \quad \text{for} \quad p > 0 \tag{A.29}$$

or

$$\int_0^2 \frac{e^{p\mu}d\mu}{\sqrt{\mu(2-\mu)}} = \pi e^{p} I_0(p) \quad \text{for} \quad p > 0 \tag{A.30}$$

Since I_0 is an even function this result essentially shows that (A.28) is valid for negative as well as positive p, while the value $p = 0$ can obviously be included for continuity reasons. (Skeptics can note that $p = 0$ gives a standard integral found in elementary tables, which is consistent with (A.28) and (A.30).)

The next step is to further generalize (A.28) by making the substitutions

$$x = \frac{2(\mu - A)}{B - A} \quad \text{and} \quad p = \frac{-\alpha(B-A)}{2} \tag{A.31}$$

where $A < B$, whereupon the integral becomes

$$e^{-\alpha A} \int_A^B \frac{e^{\alpha\mu}d\mu}{\sqrt{(\mu - A)(B-\mu)}} = \pi \exp\left(\frac{\alpha(B-A)}{2}\right) I_0\left(-\frac{\alpha(B-A)}{2}\right) \tag{A.32}$$

or

$$\int_A^B \frac{e^{\alpha\mu}d\mu}{\sqrt{(\mu - A)(B-\mu)}} = \pi \exp\left(\frac{\alpha(A+B)}{2}\right) I_0\left(-\frac{\alpha(B-A)}{2}\right) \tag{A.33}$$

Integrals containing powers of μ in the integrand can be obtained from (A.33) by differentiation with respect to α. Derivatives of the Modified Bessel Functions are

$$I_0'(z) = I_1(z), \qquad I_1'(z) = I_0(z) - \frac{I_1(z)}{z} \tag{A.34}$$

which give the result

$$\int_A^B \frac{\mu e^{\alpha\mu} d\mu}{\sqrt{(\mu - A)(B - \mu)}} = \pi \exp\left(\frac{\alpha(A+B)}{2}\right) \times \left[\left(\frac{A+B}{2}\right) I_0\left(-\frac{\alpha(B-A)}{2}\right) - \left(\frac{B-A}{2}\right) I_1\left(-\frac{\alpha(B-A)}{2}\right)\right] \tag{A.35}$$

and

$$\int_A^B \frac{\mu^2 e^{\alpha\mu} d\mu}{\sqrt{(\mu - A)(B - \mu)}} = \pi \exp\left(\frac{\alpha(A+B)}{2}\right) \left[\left(\frac{A^2+B^2}{2}\right) I_0\left(\frac{-\alpha(B-A)}{2}\right) + \left(\frac{B-A}{2\alpha} - \frac{B^2-A^2}{2}\right) I_1\left(-\frac{\alpha(B-A)}{2}\right)\right] \tag{A.36}$$

These formulas are used to obtain the expression (4.25) by setting $A = 0$, $B = \gamma^2/(p^2 + \gamma^2)$, and $\alpha = 1$.

To show that the joint density function (4.32) contains unit volume, first make the transformation (A.14), whereupon the integral over all p and q becomes

$$\int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} P_{p,q}(p,q) dq = \int_0^{\infty} R\,dR \int_0^{2\pi} P_{p,q}(p,q) d\theta = 2\pi \int_0^{\infty} P_{p,q}(p,q) R\,dR \tag{A.37}$$

Next, transform the variable of integration in (4.32) by letting

$$\alpha = \mu - X = -\frac{\gamma^2 X}{R^2 + \gamma^2} \tag{A.38}$$

and the integral becomes

$$\frac{1}{X} \int_{-X}^{0} (1 + X + \alpha)\, e^{\alpha} d\alpha \tag{A.39}$$

which is readily integrated and is equal to unity, as required.

The normalization of the expression (7.2) is carried out by the transformation (A.14), which gives

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P_{p,q}(p,q) dp\, dq = 2\gamma^2 \int_0^{\infty} \frac{h^2(R^2+\gamma^2)+1}{(R^2+\gamma^2)^2} \exp[-h^2(R^2+\gamma^2)] R\,dR \tag{A.40}$$

With the substitution

$$t = \frac{R^2 + \gamma^2}{\gamma^2} \tag{A.41}$$

this becomes

$$\int_1^{\infty} (\gamma^2 h^2 t^{-1} + t^{-2}) e^{-\gamma^2 h^2 t} dt \tag{A.42}$$

These integrals are expressible by the exponential integral functions described in [3], which in terms of (A.42) becomes

$$\gamma^2 h^2 E_1(\gamma^2 h^2) + E_2(\gamma^2 h^2) \tag{A.43}$$

There is the recursion relationship

$$E_2(z) = \mathrm{e}^{-z} - zE_1(z) \tag{A.44}$$

which allows the E_2 function of (A.43) to be expressed in terms of the E_1 function, which cancels with the first term and gives the result $\exp(-\gamma^2 h^2)$ as indicated in Chapter 7.

The variance σ_p^2 is given by (7.6), which is transformed by the substitution (A.14) giving

$$\sigma_p^2 = \frac{\gamma^2}{\pi}\int_0^{2\pi}\cos^2\theta\, d\theta \int_0^\infty R^3\left[\frac{h^2(R^2+\gamma^2)+1}{(R^2+\gamma^2)^2}\right]\exp(-h^2R^2)dR \tag{A.45}$$

With the substitution (A.41) this becomes

$$\frac{\gamma^2}{2}\mathrm{e}^{h^2\gamma^2}\int_1^\infty [h^2\gamma^2 + (1-h^2\gamma^2)t^{-1} - t^{-2}]\mathrm{e}^{-h^2\gamma^2 t}dt \tag{A.46}$$

The second and third terms of (A.46) are again evaluated in terms of the exponential integral functions to obtain

$$\sigma_p^2 = \frac{\gamma^2}{2}\mathrm{e}^{h^2\gamma^2}[\mathrm{e}^{-h^2\gamma^2} + (1-h^2\gamma^2)E_1(h^2\gamma^2) - E_2(h^2\gamma^2)] \tag{A.47}$$

And again the E_2 function is expressed in terms of the E_1 function by (A.44) to give the result indicated in (7.7). The same steps are used in the evaluation of (7.20).

Integration of (6.7) is facilitated by the substitutions of (6.9) giving

$$P_{p,q,r}(p,q,r) = \frac{\mathrm{e}^{-G}}{4\pi^2\sqrt{|\Sigma|}}\int_{-\infty}^{\infty}(r^2+s^2)\exp[-C(s+\hat{r})^2]ds \tag{A.48}$$

with $\hat{r}$ as defined in (6.9). Now, change of integration variable from s to w via the substitution

$$s + \hat{r} = w \tag{A.49}$$

gives

$$P_{p,q,r}(p,q,r) = \frac{\mathrm{e}^{-G}}{4\pi^2\sqrt{|\Sigma|}}\int_{-\infty}^{\infty}[w^2 - 2\hat{r}w + (\hat{r}^2 + r^2)]\mathrm{e}^{-Cw^2}dw \tag{A.50}$$

The middle term vanishes because it is an odd function. The result, (6.8), is obtained from any standard table of definite integrals. Then representing G in powers of r,

$$G = \left(A - \frac{B^2}{4C}\right)r^2 + \left(D - \frac{EB}{2C}\right)r + \left(F - \frac{E^2}{4C}\right) = H(r+L)^2 + M \tag{A.51}$$

defines the new parameters,

$$\begin{aligned}\Delta &= B^2 - 4AC\\ H &= A - \frac{B^2}{4C} = -\frac{\Delta}{4C}\\ L &= \Delta^{-1}(BE - 2CD)\\ M &= F - HL^2 - \frac{E^2}{4C} = F + \Delta^{-1}(AE^2 - BDE + CD^2)\end{aligned} \tag{A.52}$$

The marginal density $P_{p,q}$ then can be obtained by integrating the expression (6.8) with respect to r to obtain

$$\begin{aligned}P_{p,q}(p,q) &= \frac{(\pi C)^{-3/2}\mathrm{e}^{-M}}{8\sqrt{|\Sigma|}}\int_{-\infty}^{\infty}\left[\left(\frac{B^2}{2C}+2C\right)r^2 + \frac{EB}{C}r + \left(\frac{E^2}{2C}+1\right)\right]\exp[-H(r+L)^2]dr\\ &= \frac{(\pi C)^{-3/2}\mathrm{e}^{-M}}{8\sqrt{|\Sigma|}}\int_{-\infty}^{\infty}\left[\left(\frac{B^2}{2C}+2C\right)w^2 + N\right]\mathrm{e}^{-Hw^2}dw\end{aligned} \tag{A.53}$$

where a change of variable

$$w = r + L \tag{A.54}$$

has been made. (The term that is linear in w has been omitted since it contributes nothing to the integral.) The result is

$$P_{p,q}(p,q) = \frac{(N\Delta - B^2 - 4C^2)\mathrm{e}^{-M}}{4C\pi\Delta\sqrt{-|\Sigma|\Delta}} \tag{A.55}$$

where

$$\begin{aligned}N = \Delta^{-2}[&2B^2CD^2 + 2B^2CE^2 - 8BC^2DE + 8C^3D^2 - 8ABCDE\\ &+ 8A^2CE^2 + B^4 - 8AB^2C + 16A^2C^2]\end{aligned} \tag{A.56}$$

with Δ as defined in (6.12) and (6.14). Finally, (A.55) reduces to (6.11).

DISTRIBUTION

Copies

DOD ACTIVITIES (CONUS)

ATTN DR RABINDER MADAN 1114SE 1
CHIEF OF NAVAL RESEARCH
BALLSTON TOWER 1
800 N QUINCY ST
ARLINGTON VA 22217-5660

ATTN GEORGE SOURIS 1
ASC ENSSC
WRIGHT PATTERSON AFB OH 45433

ATTN BARRY SCHEINER AMSRL SS SG 1
PROJECT OFFICER RADAR BRANCH
ARMY RESEARCH LABORATORY
2800 POWDER MILL ROAD
ADELPHI MD 20783-1145

NON-DOD ACTIVITIES

ATTN DR GLENN M SPARKS 1
GOVERNMENT ELECTRONIC SYSTEMS
DIVISION
GENERAL ELECTRIC COMPANY
MOORESTOWN NJ 08057

ATTN JOHN DI STEFANO 1
VITRO CORPORATION
10558 CROSS FOX AVE
COLUMBUS MO 21044

ATTN PROF YAAKOV BAR SHALOM 1
ESE DEPARTMENT U 157
260 GLENBROOK RD
STORES CT 06269-3157

ATTN JOSEPH S PRIMERANO 1
PLANNING CONSULTANTS INC
PO BOX 1676
DAHLGREN VA 22448

ATTN STEVE BAUGH 1
1900 RANDOLPH SE
ALBUQUERQUE NM 87106

Copies

ATTN EDWARD PRICE 1
LON CARPENTER 1
FMC CORPORATION
1 DANUBE DR
KING GEORGE VA 22485

ATTN PHILIP O WEST 1
GEORGIA TECH RESEARCH INSTITUTE
ELECTRONIC SYSTEMS LABORATORY
CONCEPTS ANALYSIS DIVISION
ATLANTA GA 20332-0800

ATTN MELVIN BELCHER JR 1
MANAGER MISSILE DEFENSE SYSTEM
GEORGIA TECH RESEARCH INSTITUTE
SENSORS AND ELECTROMAGNETIC
APPLICATIONS LABORATORY
ATLANTA GA 30332-0800

ATTN ED KAMEN 1
SCHOOL OF ELECTRONIC AND
COMPUTER ENGINEERING
GEORGIA TECH RESEARCH INSTITUTE
ATLANTA GA 30332

ATTN SAM BLACHNER 1
HUGHES AIRCRAFT
EO EI MAIL STOP B102
P O BOX 902
EL SEGUNDO CA 90245

ATTN DR KUO CHU CHANG 1
DEPT OF SYSTEMS ENGINEERING
SCHOOL OF INFORMATION TECHNOLOGY
AND ENGINEERING
GEORGE MASON UNIVERSITY
FAIRFAX VA 22030-4444

ATTN WALTER V DIXON 1
CENTRAL SYSTEMS LABORATORY
GENERAL ELECTRIC COMPANY
BUILDING KW ROOM D214
P O BOX 8
SCHENECTADY NY 12301

DISTRIBUTION (CONTINUED)

	Copies
ATTN ARNOLD D SEIFER LOCKHEED SANDERS INC P O BOX 0868 NASHUA NH 03061-0868	1
ATTN OLIVER E DRUMMOND PHD PROFESSIONAL ENGINEER GENERAL DYNAMICS AIR DEFENSE SYSTEMS DIVISION 10900 E 4TH ST MZ 601 75 RANCHO CUCAMONGA CA 91730	1
ATTN DR J N ANDERSON	1
DR A T ALOUANI	1
DR P K RAJAN DEPARTMENT OF ELECTRICAL ENGINEERING TENNESSEE TECHNOLOGICAL UNIVERSITY TTU BOX 05004 COOKEVILLE TN 38505	1
ATTN DR PAUL KALATA ECE DEPARTMENT DREXEL UNIVERSITY PHILADELPHIA PA 19104	1
ATTN GLENN WOODARD SYSCON CORPORATION TIDEWATER DIVISION PO BOX 1480 DAHLGREN VA 22448-1480	1
DEFENSE TECHNICAL INFORMATION CENTER CAMERON STATION ALEXANDRIA VA 22304-6145	2

INTERNAL

		Copies
A20	BEUGLASS	1
A20	LUCAS	1
B05	STATON	1
B30		1
B32		1
B32	BLAIR	10
B32	CONTE	1
B32	CURRY	1
B32	GENTRY	1
B32	GROVES	10
B32	HELMICK	1
B32	LAMBERTSON	1
B32	MCNATT	1
B32	WATSON	1
B35	BAILEY	1
B35	FENNEMORE	1
B35	HARTER	1
E231		3
E282	GRAY	1
F11	DOSSETT	1
F21		1
F21	PARKER	1
F21	RYAN	1
F406		1
F41		2
F41	FONTANA	1
F41	KNICELY	1
F41	MARTIN	1
F41	TANNER	1
F42		2
F42	KLOCHAK	1
N05		1
N05	GASTON	1
N07		1
N24		1
N24	BAILEY	1
N24	BOYER	1
N24	HANSEN	1
N24	MILLER	1
N74	GIDEP	1

REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

1. AGENCY USE ONLY *(Leave blank)*	2. REPORT DATE	3. REPORT TYPE AND DATES COVERED
	October 1994	

4. TITLE AND SUBTITLE
Statistical Studies of the Monopulse Ratio

5. FUNDING NUMBERS

6. AUTHOR(S)
Gordon W. Groves and William D. Blair

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
Naval Surface Warfare Center
Dahlgren Division (Code B32)
17320 Dahlgren Road
Dahlgren, Virginia 22448-5100

8. PERFORMING ORGANIZATION REPORT NUMBER
NSWCDD/TR-94/97

9. SPONSORING/MONITORING AGENCY NAME(S) AND

10. SPONSORING/MONITORING AGENCY REPORT NUMBER

11. SUPPLEMENTARY NOTES

DISTRIBUTION STATEMENT A
Approved for public release;
Distribution Unlimited

12a. DISTRIBUTION/AVAILABILITY
~~Distribution authorized to U.S. Government agencies only; test and evaluation (October 1994). Other requests for this document must be referred to NSWCDD (Code B32), Dahlgren, VA 22448-5100.~~

12b. DISTRIBUTION CODE

13. ABSTRACT *(Maximum 200 words)*

The complex sum and difference phasers in a monopulse system provide four real quantities that are subject to random errors because of noise. Consequently, the monopulse ratio (MR), which depends on these four measured quantities, is a stochastic variable. The joint distribution of the real and imaginary parts of the MR is obtained for several cases under the assumption that the primary variables have a four-variate Gaussian distribution. The marginal distribution for the real part of the MR, or its mean, is presented in a few simple cases. Three noise models are discussed. The first assumes the noise to be uncorrelated and isotropic. The second allows real-to-real and imaginary-to-imaginary correlation between the sum and difference phasers. The third treats a general noise covariance matrix.

The variance of the MR is known to be infinite. However, density distributions with finite variance are obtained if a threshold is placed on the magnitude of the sum phaser, discarding cases lying below the threshold. The resulting variance is developed for two simple cases where the primary mean values are zero. The dependence of the joint MR density on threshold is studied.

14. SUBJECT TERMS
Monopulse Radar, Monopulse Ratio, Isotropic Noise

15. NUMBER OF PAGES
67

16. PRICE CODE

17. SECURITY CLASSIFICATION OF REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
UNCLASSIFIED	UNCLASSIFIED	UNCLASSIFIED	SAR

GENERAL INSTRUCTIONS FOR COMPLETING SF 298

The Report Documentation Page (RDP) is used in announcing and cataloging reports. It is important that this information be consistent with the rest of the report, particularly the cover and its title page. Instructions for filling in each block of the form follow. It is important to ***stay within the lines*** to meet ***optical scanning requirements***.

Block 1. Agency Use Only *(Leave blank)*.

Block 2. Report Date. Full publication date including day, month, and year, if available (e.g. 1 Jan 88). Must cite at least the year.

Block 3. Type of Report and Dates Covered. State whether report is interim, final, etc. If applicable, enter inclusive report dates (e.g. 10 Jun 87 - 30 Jun 88).

Block 4. Title and Subtitle. A title is taken from the part of the report that provides the most meaningful and complete information. When a report is prepared in more than one volume, repeat the primary title, add volume number, and include subtitle for the specific volume. On classified documents enter the title classification in parentheses.

Block 5. Funding Numbers. To include contract and grant numbers; may include program element number(s), project number(s), task number(s), and work unit number(s). Use the following labels:

C	- Contract	**PR**	- Project
G	- Grant	**TA**	- Task
PE	- Program Element	**WU**	- Work Unit Accession No.

BLOCK 6. Author(s). Name(s) of person(s) responsible for writing the report, performing the research, or credited with the content of the report. If editor or compiler, this should follow the name(s).

Block 7. Performing Organization Name(s) and address(es). Self-explanatory.

Block 8. Performing Organization Report Number. Enter the unique alphanumeric report number(s) assigned by the organization performing the report.

Block 9. Sponsoring/Monitoring Agency Name(s) and Address(es). Self-explanatory.

Block 10. Sponsoring/Monitoring Agency Report Number. *(If Known)*

Block 11. Supplementary Notes. Enter information not included elsewhere such as: Prepared in cooperation with...; Trans. of...; To be published in.... When a report is revised, include a statement whether the new report supersedes or supplements the older report.

Block 12a. Distribution/Availability Statement. Denotes public availability or limitations. Cite any availability to the public. Enter additional limitations or special markings in all capitals (e.g. NOFORN, REL, ITAR).

DOD - See DoDD 5230.24, "Distribution Statements on Technical Documents."
DOE - See authorities.
NASA - See Handbook NHB 2200.2
NTIS - Leave blank

Block 12b. Distribution Code.

DOD - Leave blank.
DOE - Enter DOE distribution categories from the Standard Distribution for Unclassified Scientific and Technical Reports.
NASA - Leave blank.
NTIS - Leave blank.

Block 13. Abstract. Include a brief ***(Maximum 200 words)*** factual summary of the most significant information contained in the report.

Block 14. Subject Terms. Keywords or phrases identifying major subjects in the report.

Block 15. Number of Pages. Enter the total number of pages.

Block 16. Price Code. Enter appropriate price code ***(NTIS only)***

Block 17.-19. Security Classifications. Self-explanatory. Enter U.S. Security Classification in accordance with U.S. Security Regulations (i.e., UNCLASSIFIED). If form contains classified information, stamp classification on the top and bottom of this page.

Block 20. Limitation of Abstract. This block must be completed to assign a limitation to the abstract. Enter either UL (unlimited or SAR (same as report). An entry in this block is necessary if the abstract is to be limited. If blank, the abstract is assumed to be unlimited.